Convex Polytopes of Permutation Matrices

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Contents

Abstract

Given a permutation group G in S_n, we can construct the set of permutation matrices of G as a set of $n \times n$ matrices with exactly one 1 per row and column, where each matrix is defined using an element of G. We can take the convex hull of these matrices, thought of as points in $\mathbf{R}^{n^2}$ space, to form the G-permutation polytope. We find the projection of this polytope from $\mathbf{R}^{n^2} \to \mathbf{R}^n$ defined by each permutation matrix X acting on a vector $a = (a_1, ..., a_n)$. We call this the G-orbit polytope. We find properties of these two polytopes for the symmetric, alternating, and dihedral groups.

Chapter 1

Introduction

1.1 Permutation groups

A *permutation* is a bijection which takes a set A to itself. A *permutation group of a set* A is a set of permutations of A that forms a group under function composition. We will be looking at groups of permutations of a nonempty, finite set A of the form $\{a_1, a_2, ..., a_n\}$. Permutations of finite sets are given by an explicit listing of each element of the domain and its corresponding functional value. For example, we define a permutation α of the set {1,2,3,4} by specifying

$$\alpha(1) = 2, \alpha(2) = 3, \alpha(3) = 1, \alpha(4) = 4.$$

A more convenient way to express this correspondence is to write α in *cyclic form*. Cyclic form is always written as a product of m-*cycles*: elements $(a_1, a_2, ..., a_m)$ where a_1 is permuted to a_2 and so on until a_m is permuted to a_1. In cyclic notation, $\alpha = (123)$, a 3-cycle. To take products of m-cycles, move from right to left from one cycle to the next, where any missing symbol is left where it is. For example, take $\beta = (321)(54)$. Start with 1 in the right cycle; however, 1 does not appear in this cycle, so (54) fixes 1. Move on to the second cycle. It tells you to move 1 to 3; (321) sends 1 to 3. Continuing in this way, the numbers $\{1, 2, 3, 4, 5\}$ are permuted to $(2, 3, 1, 5, 4)$, in order. We could just as easily have used some list of 5 elements $\{a, b, c, d, e\}$. Under the same action β, this would be permuted to (b, c, a, e, d). Two cycles are disjoint if they share no elements in common. For example, (123) and (45) are disjoint, (123) and (25) are not disjoint. Every permutation can be written as a product of disjoint cycles.

Define S_n to be the *symmetric group* of order n. The symmetric group of order n is the set of all permutations of the n-element set A. A standard counting argument shows that S_n has $n! = n(n-1)(n-2)\cdots 2 \cdot 1$ elements. Here are the elements of S_4:

$$\begin{array}{rrrrrr}
(1) & (1234) & (1324) & (14)(23) & (12)(34) & (13)(24) \\
(12) & (34) & (13) & (24) & (14) & (23) \\
(123) & (234) & (132) & (142) & (1243) & (143) \\
(1342) & (1432) & (243) & (134) & (124) & (1423)
\end{array}$$

One subset of S_n consists of all of the even permutations of n objects, which we now describe. Remember that we represented a permutation of n objects as a product of m-cycles. We can rewrite each m-cycle as a product of 2-cycles. For example, (1234)=(12)(13)(14). This decomposition is not unique, and we can even decompose to different numbers of 2-cycles. However, we always decompose to either an even or an odd number of decompositions. If a permutation can be decomposed to an even number of 2-cycles, then it is an *even permutation*. The set of even permutations forms a group. This subgroup of S_n is called A_n, the *alternating group of degree n*. For an example, look at the elements of A_4, the set of even permutations of 4 elements. Notice that exactly half of the elements of S_n are in A_n.

$$\begin{array}{rrrr}
(1) & (12)(34) & (13)(24) & (14)(23) \\
(123) & (134) & (243) & (142) \\
(132) & (234) & (124) & (143)
\end{array}$$

Another subset of S_n consists of all of the symmetries of a regular n-gon. This subset consists of the $2n$ elements of S_n which rotate or reflect some n-gon while preserving its position in space. In general, we can say $D_n = \langle \rho, \phi \mid \rho^n = \phi^2 = e,\ \rho\phi = \phi\rho^{n-1} \rangle$, where ρ is a rotation of an n-gon by $360/n$ degrees, ϕ is a reflection about a line of mirror symmetry, and e is the identity permutation, where no points are permuted. In other words, D_n is the set of all products of various powers of ρ and ϕ but we can use the relation $\phi^n = \rho^2 = (1)$. This subset actually forms a subgroup, called D_n, the *dihedral group of order 2n*. For example, take a 4-gon, commonly known as a square. We can rotate the square in increments of 90 degrees without changing the square's position in space. We can also flip the square like a pancake-horizontally, vertically, and diagonally. When we label the four corners of the square in a clockwise manner with 1,2,3,4, ρ and ϕ can be represented in cyclic notation as (1234) and (12)(34) respectively. Repeatedly combining these two actions with themselves or with each other give all of the possible elements of D_4. Here are the elements of D_4, the symmetries of the square. Note there are 8 elements:

$$(1), (12)(34), (13)(24), (24), (1234), (13), (14)(23), (1432).$$

When n is odd, we have an *odd dihedral group*. A permutation in an odd dihedral group fixes either 0 points or 1 point. This is obvious, because the rotations change all points, and the line of mirror symmetry of a regular odd sided n-gon goes through exactly one vertex. Therefore a reflection through this line fixes one point. When n is even, we have an *even dihedral group*. A permutation in an even dihedral group fixes either 0 points or 2 points. Clearly, the line of mirror symmetry of an regular even sided n-gon will contain either zero or two vertices. Thus, a reflection through this line will fix either zero or two points.

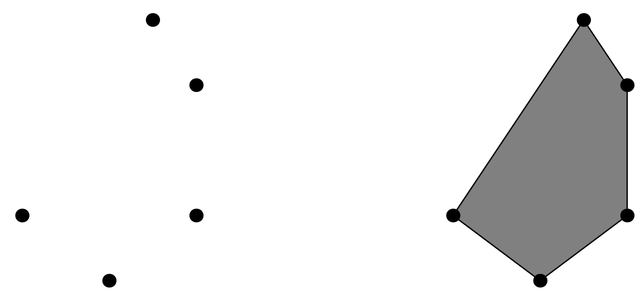

Figure 1.1: A point set and its convex hull

1.2 Polytopes

A point set is *convex* if for any two points x and y in the point set, the straight line segment

$$[x, y] = \{\lambda x + (1 - \lambda)y \mid 0 \leq \lambda \leq 1\}$$

between them is also in the point set. Every intersection of convex sets is convex. The *convex hull* of a set of points is the "smallest" convex set containing the points. Specifically, for any point set K, the convex hull of K is constructed by taking the intersection of all convex sets that contain K:

$$\mathrm{conv}(K) := \bigcap \{K' \subset \mathbf{R}^d \mid K \subset K', K' \text{ is convex}\}.$$

If K is a finite set, this convex hull will be called a *V-polytope*.

Another creation is the *H-polyhedron*, which uses the concept of halfplanes. A halfplane is just as it sounds: all of the area to one side of a defining cut; that is, those points $x \in \mathbf{R}^n$ defined by $c \cdot x \leq c_0$ for some constant c_0 and some $c \in \mathbf{R}^n$. An *H-polyhedron* P is formed by taking the intersection of finitely many closed

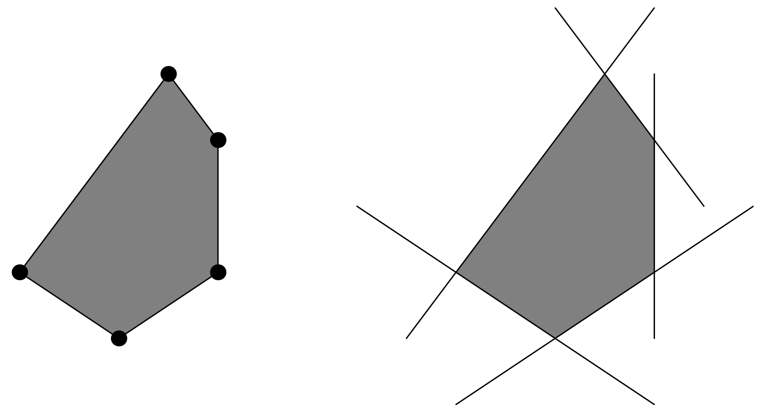

Figure 1.2: A V-polytope and an H-polytope

halfplanes in some $\mathbf{R}^d$:

$$P = P(A, z) = \{x \in \mathbf{R}^d \mid Ax \leq z\} \quad \text{for some } A \in \mathbf{R}^{m\times d}, z \in \mathbf{R}^m.$$

An *H-polyhedron* that is bounded in the sense that it does not contain a ray $\{\mathbf{x}+t\mathbf{y} \mid t \geq 0\}$ for any $y \neq 0$ is called an *H-polytope*. It turns out that every V-polytope is an H-polytope, and vice-versa (for the proof, see [Ziegler], p. 29). From now on, we will use the word *polytope* to mean V-polytope or H-polytope.

1.2.1 Faces

We will be looking at properties of *faces* of polytopes, defined to be the intersections of the polytope P with hyperplanes for which the polytope is entirely contained in one of the two halfspaces determined by the hyperplane. In other words, F is a face of P if

$$F = P \cap \{x \in \mathbf{R}^d \mid c \cdot x = c_0\}$$

where $c \cdot x \leq c_0$ is satisfied for all points $x \in P$.

To define the dimension of a face F, we first introduce the notion of the *affine hull* of F: Pick a point $p \in F$, and let L be the linear space spanned by $F - p := \{q - p \mid q \in F\}$. Then the affine hull of F, denoted aff(F), is $p + L$, i.e., the smallest affine space containing F. The dimension of aff(F) is defined to be the dimension of L. Say $v_1, ..., v_k$ is a basis for L. Thus, every point $q \in F$ can be written as $q = p + \sum a_i v_i = (1 + \sum a_i)p + \sum a_i(v_i - p_i)$. Thus we have found points $x_1 := p, x_2 := v_1 - p, ..., x_{k+1} := v_k - p$ of F such that aff(F)= $\{\sum \lambda_i x_i \mid \sum \lambda_i = 1\}$.

Definition 1.1 *The dimension of a face is the dimension of its affine hull, dim(F) := dim(aff(F)).*

In a d-dimensional polytope P, the faces of P of dimension 0 are the *vertices* of P. The *edges* of P are those faces of dimension 1. *Facets* are the $d - 1$ dimensional faces. In general, P has a set of faces of every dimension $k, 0 \leq k \leq d$. A face of *codimension* k has dimension $d - k$.

Consider the square in $\mathbf{R}^2$ created by the halfplanes $x \geq 0, y \leq 1, x \leq 1$, and $y \geq 0$. Then the vertices are the points (0,0), (1,0), (1,1), and (0,1). The edges are the intersections of the square with the lines $x = 0$, $x = 1$, $y = 0$, and $y = 1$. In this case, the edges are the facets of this polytope. The 2-dimensional face is the entire square. Two polytopes are considered *combinatorially equivalent* if there is a bijection between their faces that preserves the inclusion relation among faces. To aid the combinatorial analysis we can construct the *face lattice* of a polytope. Let S denote the set of faces of a polytope P. The inclusion relation among faces defines a partial ordering on S. Under this relation, S has a unique maximal element, namely P itself, and a unique minimal element, $\emptyset$, the empty set. Further, every two faces are minimally contained in a unique face and contain a unique maximal subface. Thus, S forms what is called a *lattice*. It turns out that if F is a k-face, then the length of any maximal totally ordered subset of S having maximal element F has

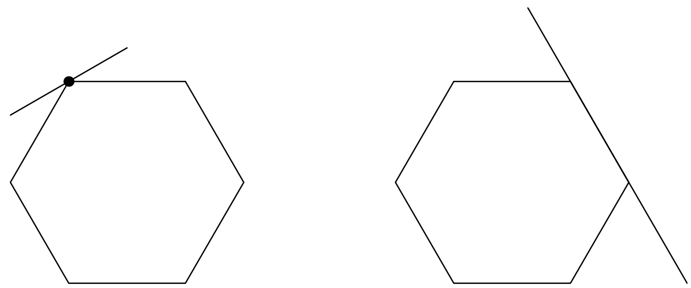

Figure 1.3: Hyperplanes defining a vertex and an edge

length $k+1$. For more on this topic, see [Ziegler]. Rephrasing what was said earlier, two polytopes are combinatorially equivalent if their face lattices are isomorphic.

1.2.2 Simplicial polytopes

The convex hull of $d+1$ affinely independent points in $\mathbf{R}^n$, where $n \geq d$, is called a *d-simplex*; thus, the d-simplex is a polytope of dimension d with $d+1$ vertices. In two dimensions, a triangle is a simplex. A tetrahedron is a three dimensional simplex. A polytope P is *simplicial* if every facet is a simplex. For example, the icosahdron is a three dimensional simplicial polytope, since each of its facets are triangles, which are simplices. Every facet of a simplicial polytope has d vertices, and every k face has $k+1$ vertices for $k \leq d-1$ (for a proof, see [Ziegler].)

1.3 Polytopes arising from permutation groups

1.3.1 Permutation matrices

Given a permutation group G inside S_n, we define the set of *permutation matrices of* G by

$$(X_\sigma)_{ij} = \begin{cases} 1 & \text{if } \sigma(i) = j \\ 0 & \text{otherwise} \end{cases}$$

for all σ in G. The $n \times n$ matrices X_σ are 0/1-matrices with exactly one 1 per row and column.

For an example of a permutation matrix, consider $\sigma = (123)$ in S_3. Then $\sigma(1) = 2$, $\sigma(2) = 3$, and $\sigma(3) = 1$. The permutation matrix associated with σ is

$$X_\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

For another example, look at $\sigma = (123)$ in S_4. Again, $\sigma(1) = 2$, $\sigma(2) = 3$, and $\sigma(3) = 1$, but now we also have $\sigma(4) = 4$, which means 4 was not affected by the permutation. Whenever a number does not appear in a permutation, it is not affected by the permutation and a 1 appears on the diagonal:

$$X_\sigma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The set of permutation matrices of the alternating group A_n consists of exactly half of all the permutation matrices: those matrices with determinant equal to $+1$. This is because we only take those matrices which can be obtained from the identity matrix with a even number of row transpositions. From linear algebra, we know that each row transposition changes the sign of the determinant. Therefore, a permutation matrix is even only if it has determinant equal to $+1$.

1.3.2 How to get a polytope in $\mathbf{R}^{n^2}$ from a permutation group

Each of the permutation matrices of the set G can be flattened, its rows listed one after another, and considered to be a point in $\mathbf{R}^{n^2}$. The convex hull of this set forms the *G-polytope*, or *permutation polytope*,

$$P(G) := \text{ conv } \{X_\sigma \mid \sigma \in G\}.$$

1.3.3 The structure of $P(G)$

In this section we will look at the vertices, edges, and facets of the permutation polytope $P(G)$.

Theorem 1.2 *Each X_σ is a vertex of $P(G)$*

PROOF Consider maximizing the inner product $< X, X_\sigma >$ as X varies over $P(G)$.

$$\begin{aligned} < X, X_\sigma > &= \sum_{1 \le i,j \le n} x(i,j) x_\sigma(i,j) \\ &= \sum_{1 \le j \le n} x(i, \sigma(i)) \le n \end{aligned}$$

with equality if and only if $X(i, \sigma(i)) = 1$ for all i. That is, $< X, X_\sigma >$ is maximal exactly when X equals X_σ. So X_σ is a vertex of the polytope. $\square$

To describe the edges of $P(G)$, we can use the following theorem, known as the *cycle-decomposition theorem*. It tells us when the line between two vertices X_σ and X_π is an edge.

Theorem 1.3 Cycle Decomposition *The line segment $\{X_\sigma, X_\pi\}$ between the vertices X_σ and X_π is an edge of the polytope constructed from the convex hull of matrices X_σ such that σ is in a group G, if and only if the cycle decomposition of $\sigma^{-1}\pi$ cannot be factored into two non-trivial parts, both of which are elements of G.*

PROOF: It suffices to show the theorem with respect to the vertices X_π and X_e, where e is the identity permutation. We need to show that the line segment between X_π and X_e is an edge of the polytope if and only if the cycle decomposition for π cannot be factored into a product of two elements of the group. If the cycle decomposition factors as $\pi = \pi_1\pi_2$ then $1/2X_e + 1/2X_\pi = 1/2X_{\pi_1} + 1/2X_{\pi_2}$. For example, take $\pi = (321)(45)$. Then $\pi_1 = (321)$ and $\pi_2 = (45)$ and we see that:

$$
\begin{aligned}
\tfrac{1}{2}X_e + \tfrac{1}{2}X_{(321)(45)} &= \tfrac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \\
&= \tfrac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \\
&= \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \\
&= \tfrac{1}{2}X_{(321)} + \tfrac{1}{2}X_{(45)}.
\end{aligned}
$$

But we know from geometry that two vertices (extreme points) u and v of a convex polytope determine an edge if and only if no point $cu+(1-c)v$ with $0 \le c \le 1$ on the line segment joining u and v can be represented as a nontrivial convex combination of two points of the polytope at least one of which does not belong to the line segment. Hence, if π can be factored into two nontrivial elements of the group, a point of the line segment between e and π can be written as a convex combination

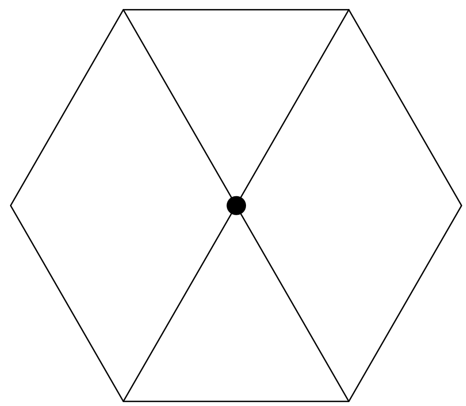

Figure 1.4: The point in the center of this polytope is not on an edge because it can be represented as a linear combination of a pair of points not on the same edge.

of two other points in the group, and so is not an edge.

If the line segment between X_e and X_π is not an edge, we will now show that the cycle decomposition for π factors nontrivially as $\pi = \sigma\tau$ where both σ and τ are in G.

Let $X = \frac{1}{2}X_e + \frac{1}{2}X_\pi$. If the line segment between X_e and X_π is not an edge, then we can write X as a positive convex combination

$$X = \sum_{\sigma \in G} \lambda_i X_\sigma, \qquad \lambda_\sigma \geq 0, \quad \sum_\sigma \lambda_\sigma = 1,$$

where some λ_σ is nonzero for $\sigma \notin \{e, \pi\}$. Fix some such σ. Since we are taking nonnegative combinations of matrices with nonnegative entries, whenever a zero appears in an entry for the matrix X, then a zero must appear in the corresponding entry in X_σ. Since there are at most two nonzero entries on each row of X, this means that if $\sigma(i) \neq i$, then $\sigma(i) = \pi(i)$.

We now show that every factor in the cycle decomposition for σ is a cycle in the cycle decompostion for π. Thus the cycle decomposition for π can be factored into two parts: σ and the product of the remaining cycles, which we denote by

τ. Since π and σ are in G and $\tau = \sigma^{-1}\pi$, it follows that τ is in G, and we will be done. To accomplish this, take i_1 such that $\sigma(i_1) \neq i_1$. The remarks in the previous paragraph show that in this case $\sigma(i_1) = \pi(i_1)$. Let $(i_1, \ldots, i_k)$ be the corresponding cycle in the cycle decomposition of π. We need to show that this cycle occurs in the decomposition for σ, as well. Suppose $\sigma(i_m) = \pi(i_m) = i_{m+1}$ for some $m < k$. By remarks in the previous paragraph, if $\sigma(i_{m+1}) \neq \pi(i_{m+1})$, then $\sigma(i_{m+1}) = i_{m+1}$. However, then we have $\sigma(i_m) = \sigma(i_{m+1})$, contradicting the fact that σ is a permutation. This completes the proof. $\square$

1.3.4 Orbits

Given a permutation group G in S_n and a point $a = (a_1, ..., a_n) \in \mathbf{R}^n$, define the *orbit* of a under G to be the set of all points

$$x_\sigma := \begin{pmatrix} a_{\sigma(1)} \\ \vdots \\ a_{\sigma(n)} \end{pmatrix} = X_\sigma \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

for $\sigma \in G$. The convex hull of the orbit defines the *orbit polytope*

$$O(G, a) = \text{ conv } \{x_\sigma \mid \sigma \in G_n\}$$

Alternatively, $O(G, a)$ is the image of $P(G)$ under the projection $\mathbf{R}^{n^2} \to \mathbf{R}^n$ defined by

$$X \mapsto X \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

thinking of $X \in \mathbf{R}^{n^2}$ as an $n \times n$ matrix.

In the following chapters, we will find that the structure of $O(G,a)$ depends on the vector a for certain groups. In general, we only know about the vertices of the projected permutation polytope.

Theorem 1.4 *Each point x_σ is a vertex of $O(G,a)$.*

PROOF Since $O(G,a)$ is the convex hull of the set x_σ, we know at least one of these points is a vertex. Pick a vertex x_σ. Let x_π be any other point in the orbit. Then $\sigma\pi^{-1}$ defines a linear isomorphism from $\mathbf{R}^n \mapsto \mathbf{R}^n$ sending $O(G,a)$ to itself and sending x_π to x_σ. Hence x_π is a vertex, too. $\square$

1.4 Summary

1.4.1 The Symmetric Group

The convex hull of the group of all $n \times n$ permutation matrices is called the Birkhoff polytope, a polytope of dimension $(n-1)^2$ with each matrix as a vertex, giving $n!$ vertices. We can describe this polytope with inequalities representing the hyperplanes which define it, (Theorem 2.1). The cycle decomposition theorem tells us how to find the edges of this polytope, (Theorem 2.3).

We proceed to take the projection of the Birkhoff polytope to get the permutahedron, which has the permutations of the vector $a = (a_1, ..., a_n)$ under permutations in S_n as its vertices. We find its dimension, (Theorem 2.4). We can describe this polytope using inequalities which define the hyperplanes framing it, (Theorem 2.7). We realize that the face lattice of the permutahedron is isomorphic to lattice of chains of subsets of $[n] := \{1, 2, ..., n\}$, (Proposition 2.10). With this information, we can determine the f-vector, which tells us how many faces there are of each dimension, and we know how to find such faces, (Theorem 2.12). Finally, we determine which vertices are adjacent to one another, finding that the vertices adjacent to a given vertex are those vertices whose coordinates differ from the given vertex by a single transposition, (Theorem 2.14).

1.4.2 The Alternating Group

In this chapter we look at the alternating polytope, an $(n-1)^2$ dimensional polytope with $n!/2$ vertices, (Theorem 3.1). The cycle decomposition theorem tell us that the line segment $\{X_\sigma, X_\pi\}$ in the alternating polytope is an edge if and only if the cycle decomposition of $\sigma^{-1}\pi$ consists of exactly 1 cycle of odd length, or exactly two cycles of even length, (Theorem 3.2). The projection of the alternating polytope yields the alternahedron, which can also be constructed by cutting vertices off of the permutahedron. We give the inequality description, and its dimension, (Theorems 3.3 and 3.4). We give data for several alternahedra and ask questions which could lead to further research.

1.4.3 The Dihedral Group

In this chapter we look at the dihedral polytope, and find that its dimension changes depending on the parity of n, (Theorem 4.1). The cycle decomposition theorem tells us that every vertex is connected to every other vertex with an edge of the polytope when $n > 4$, (Theorem 4.2). We also find that the dihedral polytopes are simplicial for odd n, (Theorem 4.3). Our data suggest several conjectures which remain to be proved.

We call the projection of the dihedral polytope the dihedron, and find its dimension, (Theorem 4.7). We also find that the dihedron is not unique for generic a. Depending on the vector we choose to permute, we can get drastically different polytopes. We look at some possibilities and ask more questions.

1.4.4 Questions

We present a list of questions which have come up in the duration of the thesis.

Chapter 2

The Symmetric Group

This chapter is mainly an exposition of theory from two sources: [Billera] and [YKK]. We have combined the ideas from both, as well as adding a few ideas of our own, to get a more complete theory than either of the others achieved alone.

The convex hull of the permutation matrices of the symmetric group, thought of as points in $\mathbf{R}^{n^2}$, notated $B_n := \text{conv}\,\{X_\sigma \mid \sigma \in S_n\}$, forms the *Birkhoff polytope*. We will now find several properties of this object.

2.1 The Birkhoff Polytope

Theorem 2.1 *B_n is an $(n-1)^2$ dimensional polytope with $n!$ vertices having the following inequality description:*

$$\begin{aligned} B_n \;\; = \;\; \{X = (x_{ij}) \in \mathbf{R}^{n^2} \mid x_{ij} \geq 0;\, 1 \leq i,\, j \leq n,\, \sum_{j=1}^{n} x_{ij} = 1 \\ \text{for } i = 1, ..., n, \;\; \text{and} \sum_{i=1}^{n} x_{ij} = 1 \;\; \text{for } j = 1, ..., n\} \end{aligned}$$

Thus, B_n consists of what are called doubly stochastic matrices: matrices with nonnegative entries and whose row and column sums are 1. We will call the right hand side of the equality C_n. Before jumping into the proof for this theorem, observe the following results about C_n:

Lemma 2.2 *The equations of C_n satisfy $2n-1$ independent linear equations.*

PROOF: In the case of $n = 3$ it is easy to represent the equations of C_n with the following matrix:

$$\left(\begin{array}{ccccccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right)$$

Subtracting the bottom three rows from the first gives

$$\left(\begin{array}{ccccccccc|c} 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right)$$

Adding rows two and three to row one gives

$$\left(\begin{array}{ccccccccc|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right)$$

Subtract columns one through three from four through six, and seven through nine, in turn, to get

$$\left(\begin{array}{ccccccccc|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

It is clear that this matrix has five linearly independent rows. This can be generalized to the matrix with $2n$ equations

$$\left(\begin{array}{ccccc|c} \vec{1} & \vec{0} & & & \vec{0} & 1 \\ \vec{0} & \vec{1} & & & \vdots & 1 \\ \vdots & \vec{0} & \ddots & & & \vdots \\ & & & \vec{1} & \vec{0} & \\ \vec{0} & \dots & & \vec{0} & \vec{1} & 1 \\ I_n & I_n & \dots & I_n & I_n & \vec{1} \end{array}\right)$$

where I_n is the $n \times n$ identity matrix, $\vec{0} = (0, ..., 0) \in \mathbf{R}^n$ and $\vec{1} = (1, ..., 1) \in \mathbf{R}^n$. Using row and column operations as before, this matrix reduces to

$$\left(\begin{array}{ccccc|c} \vec{0} & \vec{0} & \dots & \dots & \vec{0} & 0 \\ \vdots & \vec{1} & \vec{0} & & \vec{0} & 1 \\ \vdots & \vec{0} & \ddots & & \vdots & \vdots \\ \vec{0} & \vdots & & & \vec{1} & 1 \\ I_n & \vec{0} & \dots & \vec{0} & \vec{0} & \vec{1} \end{array}\right)$$

leaving us with $2n - 1$ independent linear equations. $\square$

Now we go on to prove theorem 1.1.

PROOF [BILLERA]: Remember that each X_σ is a vertex of B_n by Theorem 1.2. We know that elements of C_n satisfy $2n-1$ independent linear equations by the previous lemma; therefore, the dimension of C_n is $n^2 - (2n - 1) = (n - 1)^2$. Since each X_σ is in C_n, it follows that $B_n \subset C_n$. To show $B_n = C_n$, use induction on n to show each vertex of C_n is a permutation matrix. If a matrix X is a vertex of C_n then a standard result from the theory of polytopes says that X sits on at least $(n - 1)^2$ facets. Since the facet defining equations of C_n have the form $x_{ij} = 0$, it follows that X has at least $(n - 1)^2$ entries equal to zero. This implies that X must have a row with $n - 1$ zeroes. So, $x_{ij} = 1$ for some i and j. Without loss of generality, let $i = j = 1$. Deleting the first row and first column from X leaves us with a

$(n-1)\times(n-1)$ matrix which we will call $\tilde{X}$. We would like to show that $\tilde{X}$ is a vertex of C_{n-1}. Then, by induction $\tilde{X}$ is a permutation matrix, hence X was one, as desired. Since X is a vertex, there exists some $D=(d_{11},...,d_{nn})\in\mathbf{R}^{n^2}$ such that $\max\{\langle Y,D\rangle \mid Y\in C_n\}=X$. This implies that $\sum D_{ij}Y_{ij}\leq\sum D_{ij}X_{ij}$ for all $Y\in C_n$. Define $\tilde{D}\in\mathbf{R}^{(n-1)^2}$ by removing the first "row" and "column" of D, i.e., removing d_{1i} and d_{i1} from D for $i=1,2,...,n$. If $\tilde{X}$ were not a vertex of C_{n-1} then there would exist $\tilde{Y}$ in C_{n-1}, $\tilde{Y}\neq\tilde{X}$, such that $\langle\tilde{Y},\tilde{D}\rangle\geq\langle\tilde{X},\tilde{D}\rangle$. Define $Y\in\mathbf{R}^{n^2}$ by

$$Y:=\begin{pmatrix}1 & 0\\ 0 & \tilde{Y}\end{pmatrix}$$

It follows that $Y\in C_n, X\neq Y$, and

$$\langle Y,D\rangle=d_{11}+\langle\tilde{Y},\tilde{D}\rangle\ \geq d_{11}+\langle\tilde{X},\tilde{D}\rangle=\langle X,D\rangle.$$

which contradicts the fact that X is a vertex. Thus, $\tilde{X}$ is a vertex of C_{n-1}. It follows that X must also be a permutation matrix. □

We can use the cycle-decomposition theorem to get the following result.

Theorem 2.3 *Let X_σ and X_π be vertices of B_n corresponding to $\sigma,\pi\in S_n$. The line segment between X_σ and X_π is an edge if and only if $\sigma^{-1}\pi$ is a cycle.*

PROOF: This follows directly from the cycle decomposition theorem, Theorem 1.3. □

For example, when $n=4$, the points connected to $X_{(1)}$ are X_σ for σ being any pure cycle except (1) itself: (1234), (1324), (12), (34), (13), (24), (14), (23), (123), (234), (132), (142), (1243), (143), (1342), (1432), (243), (134), (124), and (1423). This implies that there are 20 edges containing any given vertex.

2.2 The Permutahedron

Now define the *permutahedron* $P_n\subset\mathbf{R}^n$ to be the convex hull of all permutations of the vector $(1,2,...,n)$. Specifically, in $\mathbf{R}^4$, we would have the vector $(1,2,3,4)$.

Using the symmetric group, we find all 4!=24 permutations of this vector:

(1,2,3,4)	(2,1,3,4)	(3,1,2,4)	(4,1,2,3)
(1,2,4,3)	(2,1,4,3)	(3,1,4,2)	(4,1,3,2)
(1,3,2,4)	(2,3,1,4)	(3,2,1,4)	(4,2,1,3)
(1,3,4,2)	(2,3,4,1)	(3,2,4,1)	(4,2,3,1)
(1,4,2,3)	(2,4,1,3)	(3,4,1,2)	(4,3,1,2)
(1,4,3,2)	(2,4,3,1)	(3,4,2,1)	(4,3,2,1)

It turns out that if we plot all of these points, we find that P_4 lies on a three dimensional hyperplane. This shape can be visualized by first imagining an Egyptian pyramid at the edge of a calm lake. Looking at the pyramid and its reflection as a single object, we get the octahedron. Now, imagine this octahedron enclosed in a cube just too small for it. Thus, the corners of the octahedron are cut off, leaving square faces near where the vertices of the octahedron used to belong. The faces of the octahedron which used to be triangles are now hexagons. Imagine, if you will, the Birkhoff polytope, sitting in 16-dimensional space. Remember that this polytope is made up of all of the matrices of the symmetric group of order four, each matrix being a vertex of this greater polytope. You probably cannot visualize this object, since we have a hard time thinking of objects in more than three dimensions. However, we can see its shadow. As a hand casts a shadow on a wall, the Birkhoff polytope casts a shadow on a three dimensional hyperplane, and that shadow is the permutahedron. This mathematically crude description will now be refined.

2.2.1 Face Description

More generally, let $a = (a_1, ..., a_n) \in \mathbf{R}^n$ and define the permutahedron to be:

$$P_n = \text{ conv } \{x_\sigma \mid \sigma \in S_n\}$$

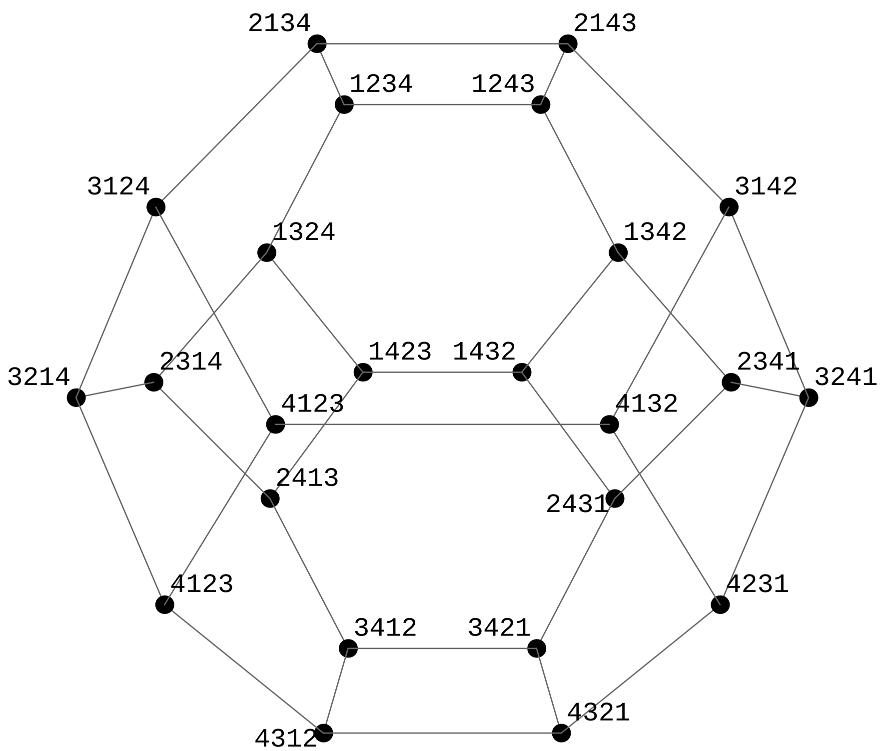

Figure 2.1: This is $P_4(1,2,3,4)$, the truncated octahedron.

where

$$x_\sigma := \begin{pmatrix} a_{\sigma(1)} \\ \vdots \\ a_{\sigma(n)} \end{pmatrix} = X_\sigma \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

for $\sigma \in S_n$.

Theorem 2.4 *If the coordinates of a are pairwise distinct, then the dimension of $P_n(a)$ is $n-1$.*

PROOF: Since $P_n(a)$ is contained in the hyperplane with equation $\sum_{i=1}^n x_i = \sum_{i=1}^n a_i$, its dimension is at most $n-1$. To see that the dimension of $P_n(a)$ is equal to $n-1$, check that $(a_1, ..., a_n)$ and the $n-1$ points obtained by transposing a_i, a_{i+1} for $i = 1, ..., n-1$ are affinely independent. □

Our next goal is Theorem 2.7, finding an inequality description for $P_n(a)$. To prove this description of $P_n(a)$, we must first learn some things about *majorizing vectors*.

Definition 2.5 The vector $x = (x_1, ..., x_n)$ *majorizes* the vector $y = (y_1, ..., y_n)$, written $x \succ y$ (we use altered notation from that used by other sources for simplicity) if, after reordering when necessary, $x_1 \leq ... \leq x_n$, $y_1 \leq ... \leq y_n$, and

$$\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$$

and

$$\sum_{i=1}^k x_i \geq \sum_{i=1}^k y_i \quad \text{for } k = 1, ..., n-1.$$

The following lemma, due to Schur, gives the necessary and sufficient conditions for the majorization of vectors. Recall that a doubly stochastic matrix is exactly an element of the Birkhoff polytope, i.e., an $n \times n$ matrix with nonnegative entries whose row and column sums are 1.

Lemma 2.6 *The vector x majorizes the vector y if and only if there is a doubly stochastic matrix Δ such that $x = \Delta y$.*

PROOF: Suppose $x_1 \leq ... \leq x_n$ and $y_1 \leq ... \leq y_n$. The proof goes by induction. Suppose $\sum_{i=1}^k x_i \geq \sum_{i=1}^k y_i$ for $k = 1, ..., n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. It follows that $\sum_{i=1}^n x_i = x_n + \sum_{i=1}^{n-1} x_i = y_n + \sum_{i=1}^{n-1} y_i$. This implies that $y_n - x_n = \sum_{i=1}^{n-1} x_i - \sum_{i=1}^{n-1} y_i \geq 0$. Therefore $y_n \geq x_n$. Then $y_n \geq x_n \geq x_1 \geq y_1$ which implies that there exists some k such that $y_{k+1} \geq x_n \geq y_k$. For this k, choose $0 \leq \lambda \leq 1$ such that $x_n = \lambda y_k + (1-\lambda) y_{k+1}$.

Define $\tilde{x} = (x_1, ..., x_{n-1})$, and $\tilde{y} = (y_1, y_2, ..., y_{k-1}, (1-\lambda)y_k + \lambda y_{k+1}, y_{k+2}, ..., y_n)$, where $0 \leq \lambda \leq 1$. Note that the components of $\tilde{y}$ are in increasing order of magnitude, i.e., $\tilde{y}_1 \leq \tilde{y}_2 \leq ... \leq \tilde{y}_{n-1}$. To see this, we just need to check

$$y_{k-1} \leq (1-\lambda)y_k + \lambda y_{k+1} \leq y_{k+2}$$

Since $x_n = \lambda y_k + (1-\lambda)y_{k+1}$ we have:

$$\tilde{y}_k = (1-\lambda)y_k + \lambda y_{k+1} = y_k + y_{k+1} - \lambda y_k - (1-\lambda)y_{k+1} = y_k + y_{k+1} - x_n.$$

Since $y_{k+1} \geq x_n$, it follows that

$$y_{k-1} \leq y_k \leq y_k + (y_{k+1} - x_n) = \tilde{y}_k$$

Also, since $y_k \leq x_n$, it follows that

$$\tilde{y}_k = y_{k+1} + (y_k - x_n) \leq y_{k+1} \leq y_{k+2}.$$

Thus, we have shown that the components of $\tilde{y}$ are increasing.

Now we want to show that $\tilde{x} \succ \tilde{y}$. Calculate:

$$\begin{aligned}
\textstyle\sum_{i=1}^{n-1} \tilde{y}_i &= y_1 + \cdots + y_{k-1} + \tilde{y}_k + y_{k+2} + \cdots + y_n \\
&= y_1 + \cdots + y_{k-1} + (y_k + y_{k+1} - x_n) + y_{k+2} + \cdots + y_n \\
&= y_1 + \cdots + y_n - x_n \\
&= x_1 + \cdots + x_n - x_n \\
&= \textstyle\sum_{i=1}^{n-1} \tilde{x}_i
\end{aligned}$$

Secondly, if $\ell < k$, we have

$$\sum_{i=1}^{\ell} \tilde{x}_i = \sum_{i=1}^{\ell} x_i \geq \sum_{i=1}^{\ell} y_i = \sum_{i=1}^{\ell} \tilde{y}_i.$$

If $\ell \geq k$, since $x_n \geq x_\ell, x \succ y$, and $\tilde{y}_k = y_k + y_{k+1} - x_n$, we have $x_1 + \cdots + x_\ell + x_n \geq x_1 + \cdots + x_{\ell+1} \geq y_1 + \cdots + y_{\ell+1}$ which implies

$$\begin{array}{rcl} \sum_{i=1}^{\ell} \tilde{x}_i = \sum_{i=1}^{\ell} x_i & \geq & y_1 + \cdots + y_{k-1} + (y_k + y_{k+1} - x_n) + y_{k+2} + \cdots + y_{\ell+1} \\ & = & \sum_{i=1}^{\ell} \tilde{y}_i. \end{array}$$

Hence, $\tilde{x} \succ \tilde{y}$.

By the inductive hypothesis, there is a $(n-1) \times (n-1)$ doubly stochastic matrix Δ such that $\tilde{x} = \Delta \tilde{y}$. In full form, this is:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} \delta_{1,1} & \cdots & \delta_{1,n-1} \\ \vdots & \ddots & \vdots \\ \delta_{n-1,1} & \cdots & \delta_{n-1,n-1} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_{k-1} \\ (1-\lambda) y_k + \lambda y_{k+1} \\ y_{k+2} \\ \vdots \\ y_n \end{pmatrix}$$

In the $n \times n$ case, we want to find the matrix which relates x and y. We find this matrix by splitting the k^{th} column of $\Delta = (\delta_{ij})$ into two and adding a final row:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} \delta_{1,1} & \cdots & (1-\lambda)\delta_{1,k} & \lambda\delta_{1,k} & \cdots & \delta_{1,n-1} \\ \vdots & & & \vdots & & \vdots \\ \delta_{n-1,1} & \cdots & (1-\lambda)\delta_{n-1,k} & \lambda\delta_{n-1,k} & \cdots & \delta_{n-1,n-1} \\ 0 & \cdots & \lambda & (1-\lambda) & \cdots & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

This new matrix is doubly stochastic. $\square$

We now have the tools necessary for proving the following theorem:

Theorem 2.7 *The permutahedron generated by the vector $a = (a_1, ..., a_n)$, with $a_1 \leq \cdots \leq a_n$, denoted $P_n(a)$, has the inequality description*

$$P_n(a) = \{x \in \mathbf{R}^n \mid \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} a_i, x(S) \geq \alpha_S \text{ for all } S \subset [n]\},$$

where $x(S) = \sum_{i \in S} x_i$, $\alpha_S = \sum_{i=1}^{|S|} a_i$, and $[n] = \{1, 2, ..., n\}$.

PROOF: Given $x \in \mathbf{R}^n$, let $\tilde{x}$ be a point in $\mathbf{R}^n$ obtained by permuting the coordinates of x so that they appear in non-decreasing order. It follows that for any $S \subset [n]$, we have $x(S) \geq \sum_{i=1}^{|S|} \tilde{x}_i$. Using this fact, one can see that the above inequalities describe the set of all points x which majorize a. Using the previous lemma, this is the set of all points of the form $x = \Delta y$ as Δ runs over all of the doubly-stochastic matrices; that is, all points in B_n. This is the projection definition of the permutahedron. Therefore, the above inequalities yield the permutahedron $P_n(a)$. □

Now that we know the inequality description of $P_n(a)$, we can go on to describe the lattice of faces of $P_n(a)$, and specifically we can determine information about the facets, vertices, and edges of $P_n(a)$. We will use the following result.

Lemma 2.8 *Let S, T be sets, and let $a = (a_1, ..., a_n) \in \mathbf{R}^n$ with $a_1 \leq \cdots \leq a_n$. Define the function $\alpha_S := \sum_{i=1}^{|S|} a_i$. Then $\alpha_S + \alpha_T \leq \alpha_{S \cap T} + \alpha_{S \cup T}$. If $a_1 < \cdots < a_n$, then the inequality becomes equality if and only if $S \subset T$ or $T \subset S$.*

PROOF: Define $|S \cap T| = u$, $|S| = u + v$, $|T| = u + w$. Then $|S \cup T| = u + v + w$. It follows that

$$\begin{aligned} \alpha_S &= a_1 + ... + a_{u+v} \\ \alpha_T &= a_1 + ... + a_{u+w} \\ \alpha_{S \cup T} &= a_1 + ... + a_{u+v+w} \\ \alpha_{S \cap T} &= a_1 + ... + a_u \end{aligned}$$

Furthermore,

$$\begin{aligned} (\alpha_{S \cap T} + \alpha_{S \cup T}) - (\alpha_S + \alpha_T) &= (\alpha_{S \cup T} - \alpha_S) - (\alpha_T - \alpha_{S \cap T}) \\ &= (a_{u+v+1} + ... + a_{u+v+w}) - (a_{u+1} + ... + a_{u+w}) \geq 0. \end{aligned}$$

If $a_1 < \cdots < a_n$ note that the last equation is equal to zero if and only if $v = 0$, that is, when $S \subset T$, or when $w = 0$, giving the trivial result of $0 = 0$ when $T \subset S$. □

Corollary 2.9 [Billera] *Let $a_1 < ... < a_n$. F is a face of $P_n(a)$ of codimension k if and only if equality in $x(S) \geq \alpha_S$ holds for precisely k distinct proper subsets lying in a chain $S_1 \subset ... \subset S_k \subset [n]$.*

PROOF If $x \in P_n(a)$ satisfies $x(S) = \alpha_S$ and $x(T) = \alpha_T$ then

$$\alpha_S + \alpha_T = x(S) + x(T) = x(S \cup T) + x(S \cap T) \geq \alpha_{S \cup T} + \alpha_{S \cap T}.$$

It follows from the previous lemma that $\alpha_S + \alpha_T = \alpha_{S \cup T} + \alpha_{S \cap T}$ and further that $S \subset T$ or $T \subset S$. Thus, equality holds in $x(S) \geq \alpha_S$ for k proper subsets S if and only if the subsets form a chain $S_1 \subset \cdots \subset S_k \subset [n]$. The resulting system of linear equations will look something like $x_1, x_1 + x_2, ..., x_1 + x_2 + ... + x_k$ which are necessarily linearly independent. $\square$

Note: From now on, we will assume $a_1 < a_2 < \cdots < a_n$.

Thus, with the above assumption, the face lattice of $P_n(a)$ is the same as as the lattice of chains of subsets in $[n]$, ordered by refinement. For an alternative description, denote by Π_n the partially ordered set of all *ordered* partitions of $[n]$, ordered by refinement. The elements of Π_n are ordered tuples $\pi = (Q_1, ..., Q_k)$ where the Q_i are pairwise disjoint subsets of $[n]$ whose union is $[n]$. Elements smaller than π have the form $(Q_{11}, ..., Q_{1j_1}, ..., Q_{k1}, ..., Q_{kj_k})$ where $(Q_{i1}, ..., Q_{ij_i})$ is an ordered partition on Q_i. For example, in Π_4, $(\{1,4\},\{3\},\{2\}) \leq (\{1,3,4\},\{2\})$. If we include in Π_n an element $\hat{0}$ such that $\hat{0} \leq \pi$ for every ordered partition π, then Π_n forms a lattice. Look at a sample lattice, Π_3:

$$\begin{array}{ccccc} & & \{1,2,3\} & & \\ & \swarrow & \downarrow & \searrow & \\ \{1,2\},\{3\} & & \{1\},\{2,3\} & & \{1,3\},\{2\} \\ & \searrow & \downarrow & \swarrow & \\ & & \{1\},\{2\},\{3\} & & \\ & & \downarrow & & \\ & & \hat{0} & & \end{array}$$

where each layer is less than the layer above it.

Proposition 2.10 *$P_n(a)$ is isomorphic to Π_n*

PROOF: Define a mapping $\tau : P_n(a) \to \Pi_n$ as follows: For a face $F \subset P_n(a)$ of codimension k, let $S_1 \subset \cdots \subset S_k \subset [n]$ be the chain given in Corollary 2.9. For notational purposes, let $S_0 = \hat{0}$ and $S_{k+1} = [n]$. Then define $\tau(F) = (Q_1, ..., Q_{k+1})$ where $Q_i := S_i \setminus S_{i-1}$. It is straightforward to check that $\tau : P_n(a) \mapsto \Pi_n$ is an isomorphism of lattices which sends a face of codimension k to a $(k+1)$-tuple in Π_n. □

Corollary 2.11 [Billera] *Faces of $P_n(a)$ are combinatorially equivalent to $P_{n_1}(a) \times ... \times P_{n_k}(a)$ where $n_1 + ... + n_k = n$.*

PROOF: Under the isomorphism defined in the previous proposition, the face lattice of a face of $P_n(a)$ is isomorphic to an interval $[\hat{0}, \pi]$ in Π_n where $\pi = (Q_1, ..., Q_k)$, using the notations from the proof of the proposition. Letting $n_i = |Q_i|$, it is easy to see that the interval $[\hat{0}, \pi]$ is isomorphic as a lattice to $\Pi_{n_1}(a) \times ... \times \Pi_{n_k}(a)$. □

Let $f = (f_0, ..., f_{n-1}) \in \mathbf{Z}^n$ where f_i is the number of faces of $P_n(a)$ of dimension i. This is called the *f-vector* of $P_n(a)$.

Theorem 2.12 [YKK] *The components of the f-vector of the permutation polytope $P_n(a)$ are given by, for all $k \in [n-1]$,*

$$f_k(P_n(a)) = \sum \frac{n!}{t_1!t_2! \cdots t_{n-k}!}$$

where the sum is carried out over all positive integral solutions of the equation $t_1 + t_2 + \cdots + t_{n-k} = n$.

PROOF: According to Corollary 2.9, faces of $P_n(a)$ of dimension k have a one-to-one correspondence with ordered partitions $(Q_1, ..., Q_{n-k})$. So f_k is given by the number of $(n-k)$-tuples $(Q_1, ..., Q_{n-k})$ where the Q_i are disjoint, non-empty, and $Q_1 \cup \cdots \cup Q_{n-k} = [n]$. The result follows from standard combinatorial analysis. □

For an example, calculate the f-vector for $P_4(a)$.

Take $n = 4$.

For $k = 0$, we write $1 + 1 + 1 + 1 = 4$ to get $f_0 = 4! = 24$.

For $k = 1$, we write $2 + 1 + 1 = 1 + 2 + 1 = 1 + 1 + 2 = 4$ to give

$$f_1 = \frac{4!}{2!1!1!} + \frac{4!}{1!2!1!} + \frac{4!}{1!1!2!} = 36.$$

For $k = 2$, we write $1 + 3 = 2 + 2 = 3 + 1 = 4$ to give

$$f_2 = \frac{4!}{1!3!} + \frac{4!}{2!2!} + \frac{4!}{3!1!} = 14.$$

For $k = 3$, we write 4=4 to give

$$f_3 = \frac{4!}{4!} = 1$$

Therefore, the f-vector is

$$f(P_4(a)) = (24, 36, 14, 1).$$

We would now like to give an explicit description of the vertices of each face of $P_n(a)$. Denote the k-face corresponding to the ordered partition $(Q_1, ..., Q_{n-k})$ by $F(Q_1, ..., Q_{n-k})$. Let $\Phi(Q_1, ..., Q_{n-k}) = \{\sigma \mid \sigma(\bigcup_{i=1}^{m} Q_i) = \{1, 2, ..., |\bigcup_{i=1}^{m} Q_i|\}\}$.

Theorem 2.13 ([YKK]) *The vertices of the k-face $F(Q_1, ..., Q_{n-k})$ are the points x_σ for all $\sigma \in \Phi(Q_1, ..., Q_{n-k})$.*

PROOF: We will first show that the vertex x_σ for $\sigma \in \Phi(Q_1, ..., Q_{n-k})$, lies in the face $F(Q_1, ..., Q_{n-k})$. For each $m \in [n-k]$, let $S_m := \bigcup_{i=1}^{m} Q_i$, then $\sigma(S_m) = \{1, 2, ..., |S_m|\}$. Therefore, $x_\sigma(S_m) = \sum_{i \in S_m} x_{\sigma(i)} = \sum_{i \in S_m} a_{\sigma(i)} = \sum_{i=1}^{|S_m|} a_i = \alpha_{S_m}$, as required.

On the other hand, if $\sigma \notin \Phi(Q_1, ..., Q_{n-k})$, choose an $m \in [n-k]$ such that $\sigma(S_m) \neq \{1, 2, ..., |S_m|\}$. We have $x_\sigma(S_m) = \sum_{i \in S_m} x_{\sigma(i)} = \sum_{i \in S_m} a_{\sigma(i)} > \sum_{i=1}^{|S_m|} a_i = \alpha_{S_m}$. The last inequality follows since $a_1 < a_2 < ... < a_m$. Since $x_\sigma(S_m) \neq \alpha_{S_m}$, we have $x_\sigma \notin F(Q_1, ..., Q_{n-k})$. $\square$

Theorem 2.14 [YKK] *The vertices of $P_n(a)$ adjacent to the vertex x_π are the vertices obtained by transposing some pair of adjacent components of x_π.*

PROOF: Suppose the line segment between x_π and x_σ forms an edge for some $\sigma \in S_n$. The edge then has the form $F(Q_1, ..., Q_{n-1})$. Since $\bigcup Q_i = [n]$ and the Q_i are pairwise disjoint, it follows that each Q_i except for exactly one, say Q_k, has one element and Q_k has two elements. Say $Q_1 = \{q_1\}, ..., Q_{k-1} = \{q_{k-1}\}, Q_k = \{q_k, q_{k+1}\}, Q_{k+1} = \{q_{k+2}\}, ..., Q_{n-1} = \{q_n\}$. Then $\Phi(Q_1, ..., Q_{n-1})$ has two elements, σ and say τ. We have $\sigma(i) = \tau(i) = q_i$ for $i = 1, ..., k-1$ and $\sigma(i) = \tau(i) = q_{i-1}$ for $i = k+2, ..., n-1$. Without loss of generality, we can take $\sigma(k) = \tau(k+1) = q_k$ and $\sigma(k+1) = \tau(k) = q_{k+1}$. Thus $x_{\sigma,i} = a_{\sigma(i)} = x_{\tau,i}$ for $i \neq k, k+1$ and $x_{\sigma,k} = x_{\tau,k+1} = a_{q_k}$ and $x_{\sigma,k+1} = x_{\tau,k} = a_{q_{k+1}}$. Thus, we get a transposition in two adjacent places. $\square$

Chapter 3

The Alternating Group

3.1 The Alternating Polytope

The convex hull in $\mathbf{R}^{n^2}$ of the set of even permutation matrices forms the *alternating polytope* E_n.

Theorem 3.1 (Brualdi) *E_n is an $(n-1)^2$ dimensional polytope with $n!/2$ vertices.*

SKETCH OF PROOF: We know that each X_σ for $\sigma \in A_n$ is a vertex by Theorem 1.2. There are $n!/2$ elements of the alternating group A_n, so E_n has $n!/2$ vertices.

The proof that E_n is $(n-1)^2$ dimensional involves showing that there exist $(n-1)^2$ even permutation matrices $P_0 = I_n, P_1, ..., P_{(n-1)^2}$ such that the set of matrices $\{P_i - P_0 \mid 1 \leq (n-1)^2\}$ is linearly independent. Please see [Brualdi] for the complete proof. $\Box$

The facet defining equations and the combinatorial structure of E_n are not known in general. However, we have the following description of the edges of E_n. We can use the cycle-decomposition theorem again to describe the edges of E_n. It tells us when the line between two vertices X_σ and X_π is an edge.

Theorem 3.2 (Brualdi) *Let σ and π be distinct permutations in A_n. Then the line $\{X_\sigma, X_\pi\}$ is an edge of E_n if and only if the cycle decomposition of $\sigma^{-1}\pi$ consists* *of exactly 1 cycle of odd length, or exactly two cycles of even length.*

PROOF: The line segment $\{X_\sigma, X_\pi\}$ is an edge of E_n if and only if $\sigma^{-1}\pi$ cannot be decomposed into two nontrivial disjoint elements of A_n, by the cycle decomposition theorem, (Theorem 1.3). This can only occur when $\sigma^{-1}\pi$ is one odd length cycle or the product of two even length cycles. □

3.2 The Alternahedron

The convex hull of all even permutations of the point $a = (a_1, ..., a_n)$, where the coordinates are pairwise distinct, is defined to be the *alternahedron*, denoted $H_n(a)$, as discussed in the first chapter. Define O_n to be the set of odd permutations: $O_n = \{\phi \mid \phi \in S_n \setminus A_n\}$. By Theorem 2.14, we know that for $\phi \in O_n$, the $n-1$ affinely independent vertices adjacent to a_ϕ, call a_{δ_i} for $i = 1, ..., n-1$, are all even. Then the unique hyperplane T_ϕ which passes through all the vertices a_{δ_i} strictly separates a_ϕ from the polytope conv $\{a_\tau \mid \tau \in S_n \setminus \phi\}$ which it supports. Thus, the intersection of the polytope $P_n(a)$ and all the half spaces $T_\sigma, \sigma \in O_n$ is precisely the polytope $H_n(a)$. The equations for these half spaces are determined in [YKK]. Setting

$$c_1 = a_1, c_2 = a_2, c_i = c_{i-1} - \frac{(a_{n-1} - a_n)(a_1 - a_2)}{a_{n-i+1} - a_{n-i+2}}$$

The desired hyperplane T_σ is given by the equation

$$\sum_{i=1}^{n} c_{\sigma(i)} x_i = \sum_{i=1}^{n} c_i a_{n-i+1} + (a_{n-1} - a_n)(a_1 - a_2).$$

Theorem 3.3 ([YKK], Theorem 3.13) *The even permutation polytope $H_n(a)$ is given by the inequalities of the permutation polytope*

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} a_i, \quad x(S) \geq \alpha_S \quad \text{for all } S \subset [n]$$

and the halfspaces

$$\sum_{i=1}^{n} c_{\phi(i)} x_i \geq \sum_{i=1}^{n} c_i a_{n-i+1} + (a_{n-1} - a_n)(a_1 - a_2) \quad \forall \phi \in O_n$$

If $n > 4$, then every inequaltity defines a face.

Since we can think of the alternadehron as the polytope contructed by cutting off half of the vertices of the permutatedron, and we know the permutahedron has $n!/2$ vertices, we know that the alternahedron has $n!/2$ vertices.

Theorem 3.4 *The alternahedron $H_n(a)$ has dimension $n-1$.*

PROOF: The points of $H_n(a)$ adjacent to an odd vertex of the full permutahedron are affinely independent. □

There are many unsolved mysteries concerning the alternahedron. We have experimental data for the first few cases; however, the alternahedron grows large very quickly, and any information above $n = 6$ takes a very long time for the computer to compile. We have the following conjecture about the alternahedron:

Conjecture 3.5 *The alternahedron $H_n(a)$ has $n!/2$ facets containing $n-1$ vertices. This says that $H_n(a)$ has $n!/2$ simplicial facets.*

For $n = 3$ and 4, these are all of the facets of the alternahedron; however, for $n > 4$, $H_n(a)$ has other facets with varying numbers of vertices on them. For example, $H_5(a)$ has 60 facets with 4 vertices, which are the facets discussed in the conjecture above. However, it also has 10 facets with 12 vertices, and 20 facets with 6 vertices. We also have information on $H_6(a)$, which has 360 facets with 5 vertices (these are the $n!/2$ simplicial facets again), as well as 20 facets with 18 vertices, 30 facets with 24 vertices, and 12 facets with 60 vertices. We have not been able to predict in general how many facets we will have, nor how many vertices are on each facet. Since we have been unable to find general theorems for the alternahedron, let us look at some examples in detail.

Using the computer program PORTA (see [PORTA]), we collected the following data for the alternahedron:

n	number of vertices	dimension	vertices on facets	number of facets	number of facets each vertex is on
3	3	2	2	3	2
4	12	3	3	20	5
5	60	4	4,6,12	90	8
6	360	5	5,18,24,60	422	10

Each generic $H_4(a)$ is combinatorially equivalent to an icosahedron. Checking by hand, we noticed that any copy of A_4 sitting inside A_5 obtained by fixing one element produced a facet of $H_5(a)$. It would be interesting to completely determine the correspondence between $H_5(a)$ and subgroups of A_5.

This information leads us to ask if there are formulae predicting the following data:

- Number of vertices on facets
- Number of facets
- Number of facets each vertex is on

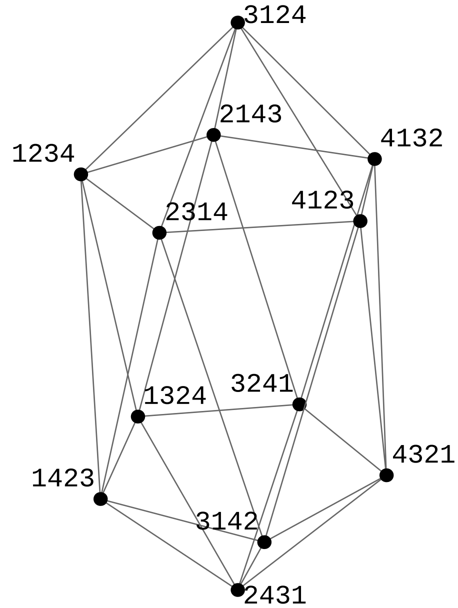

Figure 3.1: This is $H_4(1, 2, 3, 4)$, which is combinatorially equivalent to the icosahedron.

Chapter 4

Dihedral group

4.1 The Dihedral Polytope

The convex hull of the permutation matrices of the dihedral group forms the *dihedral polytope* T_n with $2n$ vertices, (Theorem 1.2).

Theorem 4.1 *The dimension of T_n is $2n-3$ when n is even, and $2n-2$ when n is odd.*

PROOF: We need to find the dimension of the linear space of hyperplanes containing T_n. To do this, form a matrix whose rows are the elements of the dihedral group, thought of in the usual way as points of $\mathbf{R}^{n^2}$. Augment the matrix by adding a final column of 1s (in order to account for the constants in the equations for the hyperplanes). Call the resulting $2n \times (n^2+1)$ matrix Z. We will show that Z has rank $2n-2$ or $2n-1$ depending on whether n is even or odd. Since elements of the kernel of Z correspond exactly with hyperplanes containing T_n, we have

$$\dim T_n = n^2 - \dim \operatorname{kernel}(T_n) = n^2 - (n^2 + 1 - \operatorname{rank}(Z)) = \operatorname{rank}(Z) - 1,$$

as required.

To explicitly construct Z, take permutations generating the dihedral group: pick $\rho = (1, 2, ..., n)$ for the rotation, and choose the reflection fixing 1, namely

$\phi = (2, n)(3, n-1)\cdots$. The first n rows of Z will correspond the rotations:

$$(1), \rho, \rho^2, \ldots, \rho^{n-1},$$

in the order listed, and the last n rows will correspond to the reflections

$$\phi, \phi\rho, \phi\rho^3, \ldots, \phi\rho^{n-1},$$

in the order listed. Now augment with a final column of 1s. Letting e_i denote the i-th standard basis vector for $\mathbf{R}^n$ allows us to write the resulting matrix as:

$$Z = \begin{pmatrix} e_1 & e_2 & e_3 & \ldots & e_n & 1 \\ e_n & e_1 & e_2 & \ldots & e_{n-1} & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ e_2 & e_3 & e_4 & \ldots & e_1 & 1 \\ e_1 & e_n & e_{n-1} & \ldots & e_2 & 1 \\ e_n & e_{n-1} & e_{n-2} & \ldots & e_1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ e_2 & e_1 & e_n & \ldots & e_3 & 1 \end{pmatrix}$$

Hence, with this notation, each column except the last represents n columns of Z. Subtracting row i from row $n+i$ for $i = 1, \ldots, n$ gives

$$\begin{pmatrix} e_1 & e_2 & e_3 & \ldots & e_n & 1 \\ e_n & e_1 & e_2 & \ldots & e_{n-1} & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ e_2 & e_3 & e_4 & \ldots & e_1 & 1 \\ \vec{0} & e_n - e_2 & e_{n-1} - e_3 & \ldots & e_2 - e_n & 0 \\ \vec{0} & e_{n-1} - e_1 & e_{n-2} - e_2 & \ldots & e_1 - e_{n-1} & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vec{0} & e_1 - e_3 & e_n - e_4 & \ldots & e_3 - e_1 & 0 \end{pmatrix}$$

Noting the shape of the first and last column, it suffices to show that the following submatrix has rank $n-2$ or $n-1$ depending on whether n is even or odd:

$$Z' = \begin{pmatrix} e_n - e_2 & e_{n-1} - e_3 & \dots & e_2 - e_n \\ e_{n-1} - e_1 & e_{n-2} - e_2 & \dots & e_1 - e_{n-1} \\ \vdots & \vdots & & \vdots \\ e_1 - e_3 & e_n - e_4 & \dots & e_3 - e_1 \end{pmatrix}$$

First we treat the even case. It is easy to check that the sum of the even-numbered rows and the sum of the odd-numbered rows are both zero. For instance, consider the sum of the odd-numbered rows of the first column of Z':

$$(e_n - e_2) + (e_{n-2} - e_n) + (e_{n-4} - e_{n-2}) + \ldots + (e_4 - e_6) + (e_2 - e_4) = 0.$$

Summing up the odd-numbered rows of each column produces a similar telescoping sum. The same argument works for the sum of the even-numbered rows. Thus, we have shown that the rank of Z' is at most $n-2$. To finish the argument in the even case, note that the following vectors from the first column of Z' are obviously linearly independent: $e_1 - e_3, e_2 - e_4, \ldots, e_{n-2} - e_n$.

We now treat the odd case. Here, it is easy to check, as in the even case, that the sum of all of the rows is zero. Hence, the rank of Z' is at most $n-1$. Again, the vectors $v_1 := e_1 - e_3, v_2 := e_2 - e_4, \ldots, v_{n-2} := e_{n-2} - e_n$ from the first column of Z' are clearly linearly independent. The vector $e_{n-1} - e_1$ also occurs in the first column. Adding the odd-numbered v_i's to $e_{n-1} - e_1$ produces the vector $v_{n-1} = e_{n-1} - e_n$. The vectors $v_1, \ldots, v_{n-1}$ are clearly linearly independent, hence, the rank of Z' in the odd case is $n-1$. □

We can again use the cycle-decomposition theorem that we used in the previous chapters to describe the edges of T_n.

Theorem 4.2 *Every vertex of T_n with $n > 4$ is connected to every other vertex of T_n by an edge of T_n.*

PROOF: Without loss of generality, examine the line segment between X_e and X_π. By the cycle decomposition theorem, (Theorem 1.3), this line segment is an edge if and only if the cycle decomposition of π cannot be factored into two parts, both of which form elements of D_n. Suppose we could factor π in such a way, say $\pi = \pi_1\pi_2$. So a fixed point in π is fixed in $\pi_1\pi_2$ which implies it is fixed in both π_1 and π_2. But, since both π_1 and π_2 are elements of D_n, at most two elements can be fixed by either of them. Thus, both π_1 and π_2 have at least $n-2$ nonfixed points. By construction, the points not fixed by π_1 are disjoint from the points not fixed by π_2. Since there are only n points altogether, we need $(n-2)+(n-2) \leq n$, which implies $n \leq 4$. □

Using the language of the proof of Theorem 4.2, the only time we can decompose π into two non-trivial parts, both of which are in D_n, is the case D_4, where $(23)(14)$ is composed of (23) and (14), which are both non-trivial elements of D_4. It follows that the line segment $\{X_{(1)}, X_{(23)(14)}\}$ is not an edge of T_4. The line segment $\{X_\sigma, X_\pi\}$ is not an edge of T_4 when $\sigma^{-1}\pi$ can be factored to $(23)(14)$. These are the only such line segments. This means that all dihedral polytopes T_n with $n > 4$ have edges connecting every pair of vertices. Thus, there are $\binom{2n}{2}$ edges. □

David Perkinson has a proof for the following theorem:

Theorem 4.3 *The odd dihedral polytopes are simplicial.*

Our data suggest the following conjectures, for which the proofs are unknown.

Conjecture 4.4 *Each facet of T_n has $2n-2$ vertices.*

The fact that odd dihedral polytopes are simplicial proves this conjecture for odd n; however, the proof for even n is unknown.

Conjecture 4.5 *The odd dihedral polytopes have n^2 facets. The even dihedral polytopes have $n^2/2$ facets.*

Conjecture 4.6 *The vertices of the odd dihedral polytopes are on $n(n-1)$ facets. The vertices of the even dihedral polytopes are on $n(n-1)/2 = \binom{n}{2}$ facets.*

These conjectures are based on computer calculations yielding the following charts.

n	number of points	dimension	vertices per facet	number of facets	number of facets each vertex is on
4	8	5	6	8	6
5	10	8	8	25	20
6	12	9	10	18	15
7	14	12	12	49	42
8	16	13	14	32	28
9	18	16	16	81	72
10	20	17	18	50	45

4.2 The Dihedron

As in previous chapters, we can take the projection of the dihedral polytope to get the dihedron, $Q_n(a)$.

Theorem 4.7 *The dihedron has dimension $n-1$.*

PROOF: Look at the subset of rotations of D_n. Consider the set of points created by these matrices acting on the vector a. For generic a, the linear space spanned by these points has dimension n by a standard result about circulent matrices ([Philip], p.75). Hence the smallest affine space containing D_n has dimension $n-1$. $\square$

Some point a is *generic* if there is an open set U about a such that $Q_n(b)$ has the same combinatorial structure as $Q_n(a)$ for all $b \in U$. Unlike the cases we examined in previous chapters, it is possible to find generic points a and a' such that $Q_n(a)$ and $Q_n(a')$ are not combinatorially equivalent. This behavior was first noted in the dihedron by David Perkinson and Douglas Squirrel in 1996. Previously, this type of behavior was noticed for more complicated groups in [Onn]. Depending on the point a we choose, we can get dramatically different polytopes. Choose polytopes $Q_5(1,2,6,4,3)$ and $Q_5(2,1,6,4,3)$. They both have the same dimension, 4, and number of vertices, 10, but the first has 35 facets with each vertex being on 14 facets, while the second has only 30 facets, with each vertex laying on 12 facets. Here is a chart of the possible number of facets from $Q_5(a)$ through $Q_8(a)$, each possibility coming from a different generic point a:

n		number of facets
5		30,35
6		20,32
7		140,154,168,182,196,210
8		118,150,190,198,222,230,246
9		612, 630, 675, 693, 738, 747, 756, 765, 774, 783, 810, 819, 828, 837,
		846, 864, 873, 891, 900, 909, 918, 927, 936, 945, 954, 963, 972,
		981, 990, 999, 1008, 1017, 1026, 1044

Notice that each possiblility for the number of facets differs from another by a multiple of n. In fact, for odd n, each possibility is a multiple of n. To get this data, we used a program called orb, created by Douglas Squirrel. The program generates random points a, checks if they are generic in the sense defined above, and then outputs the number of facets on $Q_n(a)$. Although we let the program run for some time to collect the data, it is possible that not every possibility for the number of facets appeared, especially as n increased. We predict that we would find every multiple of n, within certain bounds, in an infinite data set. It is an interesting question to ask exactly what these bounds are.

Chapter 5

Questions

In the process of writing this thesis, we found many more questions than we started with, and many of them remain unsolved. The reader may find one or more of them worth pursuing in the future. The questions are ordered by the chapter to which each relates.

Chapter Three: The Alternating Group

- Show that the alternahedron $H_n(a)$ has $n!/2$ facets containing $n-1$ vertices. This says that $H_n(a)$ has $n!/2$ simplicial facets.
- Find the complete combinatorial structure of $H_5(a)$.
- Completely describe $H_5(a)$ using relations between faces and subgroups.
- Find a formula for the number of vertices on each facet of $H_n(a)$.
- Find a formula for the number of facets of $H_n(a)$.
- Find a formula for the number of facets each vertex of $H_n(a)$ is on.

Chapter Four: The Dihedral Group.

- What are the bounds on the possible number of faces one can get from the dihedron?
- Show each facet of T_n has $2n-2$ vertices when n is even.
- Show that the odd dihedral polytopes have n^2 facets. The even dihedral polytopes have $n^2/2$ facets.
- The vertices of the odd dihedral polytopes are on $n(n-1)$ facets. The vertices of the even dihedral polytopes are on $n(n-1)/2 = \binom{n}{2}$ facets.
- What do dihedral face lattices look like?

Bibliography

[Billera] Louis J. Billera and A. Sarangarajan, *The Combinatorics of Permutation Polytopes*, DIMACS vol. 24, 1994, pp. 1–23.

[Branko] Branko Grünbaum, *Convex Polytopes*, Pure and Applied Mathematics vol. 16, Interscience Publishers, 1967.

[BGI] Brualdi, Gibson, *Convex Polyhedra of Doubly Stochastic Matrices, I* Journal of Combinatorial Theory, Series A 22, Academic Press, Inc., 1977, pp. 175–198

[BGII] ______, *Convex Polyhedra of Doubly Stochastic Matrices, II* Journal of Combinatorial Theory, Series B 22, Academic Press, Inc., 1977, pp. 338–351

[Brualdi] R. A. Brualdi, B. Liu, *Even Doubly Stochastic Matrices*, Journal of Combinatorial Theory Series A 57, 1991, pp. 247–253

[PORTA] Thomas Christof, *PORTA- A Polyhedron Representation Transition Algorithm, version 1.2.1*, revised by A. Loebel and M. Stoer, available from the ZIB electronic library eLib via elib@zib-berlin.de or anonymous ftp from elib.zib-berlin.de, dir "/pub/mathprog"

[Philip] Philip J. Davis, *Circulant Matrices*, Pure and Applied Mathematics, Interscience Publishers, 1979

[Gallian] Joseph A. Gallian, *Contemporary Abstract Algebra*, D. C. Heath and Co., Third Edition, 1994

[Hardy] G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities* Cambridge University Press, 1934

[Onn] Shmuel Onn, *Geometry, Complexity, and Combinatorics of Permutaions Polytopes*, Journal of Combinatorial Theory, Series A 64, Academic Press, Inc., 1993, pp. 31–49

[Ridenour] Judy M. Ridenour, *Combinatorics of Orbitopes*, Reed College Thesis, 1996.

[Sagan] Bruce E. Sagan, *The Symmetric Group*, Wadsworth and Brooks, Cole Publishing, 1967.

[YKK] Yemelichev, Kovalev, Kravtsov, *Polytopes, Graphs, and Optimisation*, Cambridge University Press, 1984

[Ziegler] Günter M. Ziegler, *Lectures on Polytopes*, Graduate Texts in Mathematics vol. 152, Springer-Verlag, 1995.