Critical Groups of Simplicial Complexes

A Thesis
Presented to
The Division of Mathematics and Natural Sciences
Reed College

In Partial Fulfillment
of the Requirements for the Degree
Bachelor of Arts

Marcus P. Robinson

May 2013

Approved for the Division
(Mathematics)

David Perkinson

Acknowledgments

Ben Corner, Jim Fix, David Perkinson, Jamie Pommersheim, Paul Robinson, Susan Robinson, Irena Swanson, Darko Trifunovski.

Table of Contents

Abstract

In this thesis we explore critical groups of simplicial complexes. We review the relationship between critical groups and homology groups. Our main results extend the definition of harmonic morphisms on graphs to harmonic morphisms on simplicial complexes. We show that these mappings induce group homomorphisms on the associated critical groups. In chapter three we explore critical groups from the perspective of algebraic geometry and develop a pullback that induces a mapping of critical groups.

Introduction

Figure 1 is a picture of a simplicial complex, augmented with some arrows and numbers along the edges. Imagine that these arrows (and their corresponding weights) correspond to amounts of flow along the edges in the direction indicated. The net flow across some vertex, can be calculated by summing the amount of flow coming in and subtracting the amount coming out. For example, there is a single unit flow going into the vertex labeled a and a single unit of flow leaving for a net flow of zero.

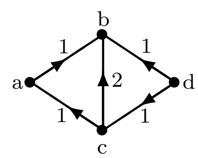

Figure 1: Simplicial complex with flow along edges

Now consider (a) in Figure 2, formed by reducing the flow along the edge from a to b, increasing the flow along the edge from a to c (here this makes the flow zero since before the flow was from vertex c to vertex a), and increasing the flow along the edge from c to b. Note that the flow configuration has changed but the net flow across any vertex is unchanged. Thus, in effect, a unit of flow has been diverted across the triangle that contains a, b and c.

If the unit of flow that was added along the edge from c to b is then diverted again, this time across the triangle containing b, c and d, the configuration shown in Figure 2 (b) is obtained. It is simple to check that this change also has had no effect on the net flow at each vertex.

Figure 2: Flow configuration after diverting across faces

This leaves us with a fairly simple game—we have some flow configuration and we can maintain the net flow across vertices by diverting across triangles. The goal of this thesis is to explore the remarkably rich structure that comes from studying this model in arbitrary dimensions.

Chapter one formally introduces the concepts outlined above. We start by describing abstract simplicial complexes and their associated homology groups. We then describe the game above in the case of one-dimensional simplicial complexes (graphs) and use this to motivate our algebraic description of critical groups. The chapter concludes by describing the relationship between homology groups and critical groups.

At this point we will have associated an algebraic object, the critical group to our simplicial complex. From here we can ask when mappings from one simplicial complex to another also gives us mappings between their respective critical groups. In chapter two, we describe a type of morphism between simplicial complex that induces a group homomorphism. We also introduce two categorical product for simplicial complexes, the first for general simplicial morphisms and the second for dimension preserving simplicial morphisms.

In the final chapter, we attempted to view critical groups as groups of algebraic cycles modulo rational equivalence. The chapter starts by introducing some algebraic geometry. The second portion of the chapter tries to mimic the constructions from the first section for critical groups. The chapter concludes by describing where the analogy breaks down.

The bulk of this text assumes only a working knowledge of linear algebra and some abstract algebra.

Chapter 1

Critical Groups

1.1 Simplicial Complexes

Our first task is to formally introduce simplicial complexes and their associated homology groups. This section contains a brief overview of these concepts. A more complete treatment can be found in [5].

1.1.1 Simplicial Homology

Definition 1.1.1. Let Σ be a finite, nonempty collection of subsets of some universal set S. The set Σ is called an *abstract simplicial complex* if for each set $X \in \Sigma$, every subset $Y \subset X$ is also in Σ.

The elements of Σ are called faces. The faces of Σ that are maximal, meaning not contained in some other face, are called *facets*. If $\sigma \in \Sigma$ we denote $\dim(\sigma) = |\sigma|$. If $\dim(\sigma) = i + 1$, then we say σ is an *i-dimensional* face of Σ . We will denote the set of all i-dimensional faces of Σ by $F_i(\Sigma) := \{\sigma \in \Sigma : \dim(\sigma) = i\}$. By convention the empty set, $\emptyset$, is the unique face of dimension -1. We will denote the number of i-faces by $|F_i(\Sigma)|$. The *dimension* of a simplicial complex Σ, denoted $\dim(\Sigma)$, is equal to the dimension of its highest dimensional face. If each facet of Σ has dimension d then we say that Σ is a *pure d-dimensional* simplicial complex.

In general, we take $S = [n] = \{1, \ldots, n\}$ as our universal set. Our notion of an abstract simplicial complex has a geometric interpretation. Informally, a simplex is a n-dimensional triangle. A simplicial complex Σ is simplices of arbitrary dimension glued together in such a way that any face of a simplex is also a face in Σ and the intersection of any two simplices in Σ is a face of both of those simplices. For our purposes, the abstract simplicial complex (referred to from now on simply as a simplicial complex), will suffice.

A few remarks on notation: Since a simplicial complex is closed under taking subsets, we will usually describe a simplicial complex by listing its facets. We will also sometimes refer to the faces of dimension zero, $F_0(\Sigma)$, as vertices and faces of dimension one, $F_1(\Sigma)$, as edges. Frequently we will be interested in substructures of our simplicial complex having at most dimension i, we will refer to this as the *i-skeleton* of Σ. For instance we can form a graph by taking the one-skeleton of any simplicial complex.

Example 1.1.2. Let $S = [4]$, and let Σ be a simplicial complex with facets

$$\Sigma = \{\{123\}, \{124\}, \{134\}, \{234\}\}.$$

Then Σ can be pictured as in Figure 1.1. Note that in this picture the 2-dimensional faces (triangles) are filled in but the tetrahedron itself is not.

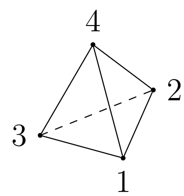

Figure 1.1: Simplicial Complex

For the rest of the section, fix R to be a commutative ring with identity 1.

Definition 1.1.3. For any finite set X, define the *free R-module* on X by

$$RX = \left\{ \sum_{x \in X} a_x x : a_x \in R \right\}.$$

Definition 1.1.4. Let Σ be a simplicial complex. An *orientation* on Σ is a partial ordering of the vertices such that for any simplex the vertices are totally ordered.

For our universal set $S = [n]$ there is a standard orientation which we will impose on all simplicial complexes for the remainder of the thesis.

Example 1.1.5. Let Σ be the simplicial complex in Figure 1.1. Then the edge going from vertex 1 to vertex 2 is

Definition 1.1.6. Let $\partial_i : RF_i(\Sigma) \to RF_{i-1}(\Sigma)$ be defined by

$$\partial_i(\sigma) := \sum_{j \in \sigma} \operatorname{sign}(j, \sigma)(\sigma \setminus \{j\})$$

where $\operatorname{sign}(j, \sigma) = (-1)^{i-1}$ if j is the ith element of σ when the elements of σ are in an increasing order and where $\sigma \setminus j := \sigma \setminus \{j\}$. Define $\partial_i^* : RF_i(\Sigma) \to RF_{i+1}(\Sigma)$ to be the dual of ∂_i.

The following proposition is standard and straightforward to prove:

Proposition 1.1.7. *For any* $\sigma \in RF_{i-1}(\Sigma)$

$$(\partial_{i-1} \circ \partial_i)(\sigma) = 0.$$

The previous proposition allows us to form a sequence.

Definition 1.1.8. Let Σ be a simplicial complex and let $\dim(\Sigma) = n$. Then the *augmented chain complex* for Σ is the sequence

$$0 \to RF_{n-1}(\Sigma) \xrightarrow{\partial_{n-1}} \cdots \to RF_i(\Sigma) \xrightarrow{\partial_i} RF_{i-1}(\Sigma) \to \cdots \xrightarrow{\partial_0} RF_{-1}(\Sigma) \to 0.$$

Definition 1.1.9. Let R be a ring. The ith-*reduced homology* group of Σ with coefficients in the ring R is defined to be

$$\overline{H}_i(\Sigma; R) := \frac{\ker \partial_i}{\text{im } \partial_{i+1}}$$

Sometimes we will simply write $\overline{H}_i(\Sigma)$ when it is clear what ring we are working over. The i*th-Betti number* of Σ is defined to be

$$\beta_i(\Sigma) := \dim_{\mathbb{Q}} \overline{H}_i(\Sigma; \mathbb{Q}).$$

We will usually fix R to be $\mathbb{Z}$ or $\mathbb{Q}$. We will conclude this section by showing an example of computing the homology for a simplicial complex.

Example 1.1.10. Let Σ be the simplicial complex in Figure 1.1. The augmented chain complex for Σ is

$$0 \xrightarrow[\partial_3]{} \mathbb{Q}F_2(\Sigma) \xrightarrow[\partial_2]{\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}} \mathbb{Q}F_1(\Sigma) \xrightarrow[\partial_1]{\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix}} \mathbb{Q}F_0(\Sigma) \xrightarrow[\partial_0]{\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}} \mathbb{Q} \to 0.$$

Then we can compute that

$$\overline{H}_0(\Sigma; \mathbb{Q}) = 0 \qquad \overline{H}_1(\Sigma; \mathbb{Q}) = 0 \qquad \overline{H}_2(\Sigma; \mathbb{Q}) = \mathbb{Q} \qquad \overline{H}_{i>2}(\Sigma; \mathbb{Q}) = 0.$$

What happens if the tetrahedron is filled in? Then Σ is the simplicial complex with a single facet $\{1, 2, 3, 4\}$. Recomputing homologies shows that

$$\overline{H}_i(\Sigma; \mathbb{Q}) = 0 \text{ for all } i \geq 0.$$

Remark 1.1.11. There are a variety of ways to think about simplicial homology. One notion is that the homology $\overline{H}_i(\Sigma)$ counts the number of $(i+1)$-dimensional holes in Σ. The previous example demonstrates this nicely by showing that the homology is zero when the tetrahedron is filled in and $\mathbb{Z}$ when it is not.

The elements of $\ker \partial_i$ are called i-cycles and the elements of $\text{im } \partial_{i+1}$ are called i-boundaries. By Proposition 1.1.6, each i-boundary is an i-cycle. We say the simplicial complex is exact at $RF_i(\Sigma)$ if every i-cycle is an i-boundary, i.e, if $\text{im } \partial_{i+1} = \ker \partial_i$.

1.1.2 Simplicial Spanning Trees

We will develop one last piece of machinery, the simplicial spanning tree, before defining critical groups. These spanning trees will play a role in how we describe the critical group and how we can relate them to homology groups.

Definition 1.1.12. Let Σ be a pure d-dimensional simplicial complex. And let $\Upsilon \subset \Sigma$ be a subcomplex such that the $d-1$-skeleton of Υ is that same as the $d-1$ skeleton of Σ. Then Υ is an *d-simplicial spanning tree* of Σ if

1. $\overline{H}_d(\Upsilon;\mathbb{Z}) = 0$,
2. $\overline{H}_{d-1}(\Upsilon;\mathbb{Q}) = 0$, and
3. $|F_d(\Upsilon)| = |F_d(\Sigma)| - \beta_d(\Sigma) + \beta_{d-1}(\Sigma)$.

In general an i-dimensional spanning tree of Σ is a spanning tree of the i-dimensional skeleton of Σ.

If our simplicial complex is a graph ($d = 1$), we recover the usual definition of a spanning tree for a graph. Since $\overline{H_1}(\Upsilon;\mathbb{Z}) = 0$ the spanning tree must be acyclic, $\overline{H}_0(\Upsilon;\mathbb{Q}) = 0$ implies that Υ is connected and

$$\begin{aligned} |F_1(\Upsilon)| &= |F_1(\Sigma)| - \beta_d(\Sigma) + \beta_0(\Sigma) \\ &= |E| - (|E| - |V| + 1) \\ &= |V| - 1, \end{aligned}$$

means that Υ contains one more vertex than edge.

Example 1.1.13. Let Σ be the simplicial complex in Figure 1.1. We can form a 2-simplicial spanning tree by choosing any three facets. It follows that there are four distinct spanning trees. We could also construct 16 different 1-simplicial spanning trees.

Definition 1.1.14. Let Σ be a simplicial complex and let Υ be an i-simplicial spanning tree. Define the *Laplacian* $\mathcal{L} : \mathbb{Z}F_i \to \mathbb{Z}F_i$ by

$$\mathcal{L}_i := \partial_{i+1}\partial_{i+1}^*.$$

Let Θ be the set of i-faces in $F_i(\Sigma)$ but not in $F_i(\Upsilon)$. Define the *reduced Laplacian* $\tilde{\mathcal{L}} : \mathbb{Z}\Theta \to \mathbb{Z}\Theta$ by removing the rows and columns corresponding to i-faces in Υ from the full Laplacian. Sometime we will also denote Θ by $\tilde{F}_i(\Sigma)$.

1.2 Critical Groups

With a handle on homology, we can now introduce our other main object of interest, the critical group. We will motivate our study of critical groups by looking at the abelian sandpile model which is is similar to the game played in the introduction. The definitions and theorems in this section come from [6] and a more complete account of the abelian sandpile model can be found there.

1.2.1 Sandpile Groups

To introduce the concept of critical groups of simplicial complexes we will look at what is known as the Abelian Sandpile Model. Before proceeding let us clarify some notation. Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E. We allow multiple edges between the same pair of vertices and define the weight function

$$\mathrm{wt}(v, w) := \text{the number of edges between } v \text{ and } w.$$

For $v \in V$ we define

$$\deg(\mathrm{v}) := \sum_{w \in V} \mathrm{wt}(v, w).$$

Definition 1.2.1. A *sandpile graph* is a pair (G,s) where G is a connected graph and $s \in V$. We refer to s as the *sink*. We denote $V \setminus s$ by $\tilde{V}$.

At the start of the game, each vertex, except the sink, has some nonnegative number grains of sand placed on it.

Definition 1.2.2. A configuration on a graph is an element in $\mathbb{Z}V$. A *configuration* on a sandpile graph G is an element $c \in \mathbb{Z}\tilde{V}$. For a configuration $c = \sum_{v \in \tilde{V}} c_v v$, a vertex $v \in \tilde{V}$ is *stable* if $c_v < d_v = \deg(v)$, otherwise it is *unstable*. A configuration is *stable* if for all $v \in \tilde{V}$, the vertex v is stable.

Each turn a non-sink vertex *fires* by sending a chip to each of its neighbors. That is, when vertex i fires, it loses a number of grains of sand equal to it degree, and each of its neighbors gain a single grain of sand. We can describe firing a vertex formally using a familiar object.

Definition 1.2.3. Let G be a sandpile graph. The Laplacian of G is the mapping $\mathcal{L} : \mathbb{Z}V \to \mathbb{Z}V$ given by

$$\mathcal{L}(v) := \deg(v)v - \sum_{u \in V} \mathrm{wt}(v, u)u.$$

The reduced Laplacian of G is the mapping $\mathcal{L} : \mathbb{Z}\tilde{V} \to \mathbb{Z}\tilde{V}$ is given by

$$\tilde{\mathcal{L}}(v) := \deg(v)v - \sum_{u \in \tilde{V}} \mathrm{wt}(v, u)u.$$

Example 1.2.4. Using definition 1.2.3 we calculate that the reduced Laplacian of the sandpile graph in Figure 1.1 is

$$\tilde{\mathcal{L}} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}.$$

Let us calculate the reduced Laplacian using the method from section 1.1.1. As noted earlier, a graph is a one dimensional simplicial complex. We can form the augmented chain complex for an arbitrary graph by

$$0 \to \mathbb{Z}E \xrightarrow{\partial_1} \mathbb{Z}V \xrightarrow{\partial_0} \mathbb{Z} \to 0.$$

Using the boundary mappings we calculated earlier we get that

$$\partial_1 \circ \partial_1^* = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}.$$

Removing the row and column associated with the sink vertex returns the reduced Laplacian which is the same matrix that we calculated previously.

In fact, we can show in general that the two definitions we have given for the Laplacian are the same

$$\begin{aligned}(\partial_1 \circ \partial_1^*)(v) &= \sum_{\substack{e \in V(G) \\ v \in e}} \partial_1(e) \\ &= \deg(v)v - \sum_{(u,v)=e \in E} \mathrm{wt}(v,u)u \\ &= \mathcal{L}(v).\end{aligned}$$

It follows that firing a vertex is the same as subtracting the associated column of the Laplacian from the current configuration vector.

Intuitively, we would like to avoid having a negative quantity of sand, so we will add restrictions about when a vertex is allowed to fire.

Definition 1.2.5. Let c be a configuration on the sandpile graph (G, s). Let $v_1, \dots, v_k$ be a sequence of nonsink vertices. We denote the configuration after firing each vertex by $c(v_1, \dots, v_k)$. A firing sequence is *valid* if v_1 is unstable and $v_i \neq s$ is unstable after firing vertices $v_1, \dots, v_{i-1})$ for $i = 2, \dots, k$.

It is natural to wonder if every configuration will eventually stabilize if enough unstable vertices are fired. Furthermore, if a configuration does eventually stabalize, is the configuration reached the same no matter what order the vertices are fired in? The answer to both questions is yes.

Theorem 1.2.6. *([6]) Every configuration c on G has a unique stabilization. We denote the stabilization of c by $(c)^\circ$.*

Definition 1.2.7. We define a binary operation called *stable addition* and denoted $\oplus$ by

$$a \oplus b = (a+b)^\circ.$$

Definition 1.2.8. Let $r \in \mathbb{N}\tilde{V}$ be stable. We say that r is *critical* if for each $t \in \mathbb{N}\tilde{V}$ there exists $s \in \mathbb{N}\tilde{V}$ such that

$$(s+t)^\circ = r.$$

This leads us to the main result of the section.

Theorem 1.2.9. *The critical elements of G form a group under $\oplus$, denoted $S(G)$, called the sandpile group.*

To illustrate the concepts from this section we give an explicit example of a sandpile group.

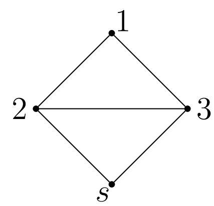

Figure 1.2: Sandpile Graph

Example 1.2.10. Let G be the graph in Figure 1.2.10. The critical configurations are

$$\begin{array}{cccc} (2,2,1) & (2,2,0) & (1,2,0) & (2,0,1) \\ (0,2,1) & (2,1,0) & (1,2,1) & (2,1,1). \end{array}$$

Let us calculate a few stable additions

$$\begin{aligned} ((2,2,0) \oplus (2,2,0))^{\circ} &= (2,2,0) \\ ((1,2,0) \oplus (2,2,0))^{\circ} &= (1,2,0). \end{aligned}$$

Tedious legwork would demonstrate that these configurations do form a group under stable addition. More work shows that $S(G) \cong \mathbb{Z}_8$.

Before moving on, it is worth looking at the results of this section from an algebraic perspective. Suppose that G has n vertices not including the sink. It is clear that $(\partial_0 \circ \tilde{L})(v) = 0$ so we can write a new chain complex

$$\mathbb{Z}V \xrightarrow{\mathcal{L}} \mathbb{Z}V \xrightarrow{\partial_0} \mathbb{Z} \to 0.$$

It easy to check that rank $(\mathcal{L}) = n = \operatorname{rank}\ker\partial_0$. It follows that $\ker(\partial_0)/\mathrm{im}\ \mathcal{L}$ is a finite abelian group.

Definition 1.2.11. Let Σ be a simplicial complex. Then the zero-order critical group of Σ is

$$K_0(\Sigma) := \ker\partial_0/\mathrm{im}\ \mathcal{L}.$$

Our last order of business is to establish a relation between the sandpile group and the zero order critical group.

Theorem 1.2.12. *Let $c = \sum_{v\in\tilde{V}} c_v v \in \mathbb{N}\tilde{V}$ and define $\tilde{c} = c - (\sum_{v\in\tilde{V}} c_v)s \in \mathbb{Z}V$. There are isomorphisms of abelian groups*

$$\begin{array}{lll} S(G) & \to \mathbb{Z}\tilde{V}/\text{im } \tilde{\mathcal{L}} & \to K_0(G) \\ c & \mapsto c+\tilde{\mathcal{L}} & \mapsto \tilde{c}+\tilde{\mathcal{L}}. \end{array}$$

Thus, each element of $\mathbb{Z}\tilde{V}$ is equivalent to a unique element modulo the image of the reduced Laplacian. The second isomorphism shows that up to isomorphism, the sandpile group does not depend on the choice of the sink vertex.

1.2.2 Higher Order Critical Groups

The notion of critical groups for simplicial complexes was developed in [3]. We summarize some of their theorems and definitions here.

The setting for this section will be a simplicial complex Σ with augmented chain complex and boundary maps described as above. We can now construct a chain complex in analogy to the one created in section 1.2.1 using the Laplacian

$$\mathbb{Z}F_i \xrightarrow{\mathcal{L}_i} \mathbb{Z}F_i \xrightarrow{\partial_i} \mathbb{Z}F_{i-1}.$$

Definition 1.2.13. The ith critical group, denoted $K_i(\Sigma)$, is given by

$$K_i(\Sigma) := \frac{\ker\, \partial_i}{\text{im } \mathcal{L}_i} = \frac{\ker\, \partial_i}{\text{im } \partial_{i+1}\partial_{i+1}^*}.$$

We can now state the main result of the chapter.

Theorem 1.2.14. *([3]) Let Σ be a simplicial complex and let Υ be an i-simplicial spanning tree. Let $\Theta = F_i(\Sigma) \setminus F_i(\Upsilon)$. Then if $\overline{H}_{i-1}(\Upsilon;\mathbb{Z}) = 0$,*

$$K_i(\Sigma) = \mathbb{Z}^{\Theta}/im\ \tilde{\mathcal{L}}.$$

Note that here we require $\overline{H}_{i-1}(\Upsilon;\mathbb{Z}) = 0$, not just that $\overline{H}_{i-1}(\Upsilon;\mathbb{Q}) = 0$ as required for an i-simplicial spanning tree.

To clarify the previous result we offer the following example.

Example 1.2.15. Let Σ be the simplicial complex in Figure 1.1. A 1-simplicial spanning tree can be formed using the faces $f_{123}, f_{124}, f_{134}$. We calculate that the reduced Laplacian is

$$\tilde{\mathcal{L}}_i = \begin{array}{c} \\ 23 \\ 24 \\ 34 \end{array} \begin{array}{c} \begin{array}{ccc} 23 & 24 & 34 \end{array} \\ \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & -2 \end{pmatrix} \end{array}.$$

Then $K_1(\Sigma) = \mathbb{Z}F_1(\Sigma)/\text{im } \tilde{\mathcal{L}}_1 = \mathbb{Z}_4$. A similar method gives that $K_0(\Sigma) = \mathbb{Z}_4 \times \mathbb{Z}_4$.

1.2.3 Critical Groups as a Model of Discrete Flow

Recall that we defined a configuration vector on a sandpile graph to be $c \in \mathbb{Z}\tilde{V}$ and a configuration on the entire graph to be $c \in \mathbb{Z}V$. For an arbitrary simplicial complex we define a configuration of i-faces as $c \in \mathbb{Z}F_i(\Sigma)$. Each element of the critical group $K_i(\Sigma)$ can be represented by a configuration vector modulo the equivalence relation given by the Laplacian.

Let us examine the one-skeleton of a simplicial complex Σ. There is a natural notion of interpreting a configuration vector on the 1-faces of Σ as flow along the edges. A positive value would indicate flow in the direction that the edge is oriented and a negative value would indicate flow in the opposite direction of the orientation. In general we can think of a configuration vector c describing i-flow with the understanding that a positive value means in the direction of the orientation and a negative value indicates flow in the direction opposite the orientation.

The boundary map ∂_i naturally converts a configuration vector (or in our new terminology an i-flow) into an $(i-1)$-flow. In the case of graphs, firing a vertex meant that no grains of sand were created or lost. With $i = 2$, firing leaves the net flow into a vertex unchanged. If c is in $\ker \partial_i$ then the sum of the $(i-1)$-flows resulting from applying the boundary map ∂_i must cancel.

Configurations c and c' in critical groups are equivalent if there exists some linear combination of columns from the Laplacian that we can add to c to form c'. The analogous concept that we developed for sandpile graphs was that two configurations were equivalent if there existed a firing sequence that moved us from one configuration to the other. We can develop a similar notion of firing an i-face.

In the case of $i = 1$, firing an edge (subtracting the corresponding column vector to the configuration vector) means that we divert a unit of flow around each 2-face that the edges sits on. In general firing an i-face means diverting a unit of flow around each $i+1$-face that the i-face sits on. The next example gives a visual demonstration of this.

Example 1.2.16. Let Σ be the simplicial complex pictured in Figure 1.2 (a) with the faces $\{123\}, \{234\}$ included. The Laplacian for Σ is given by

$$\mathcal{L}_1 = \begin{array}{c} \\ 12 \\ 13 \\ 23 \\ 24 \\ 34 \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} 12 & 13 & 23 & 24 & 34 \end{array} \\ \begin{pmatrix} 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 \\ 1 & -1 & 2 & -1 & 1 \\ 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix} \end{array}.$$

Suppose that we are in the configuration pictured in (b) and fired edge (23). With our notion of conservative flow, this is the same as diverting a unit of flow around the face (123) and the face (234). The result of this firing is pictured in (c). Firing edge (12) in the configuration pictured in (b) results in the configuration pictured in (d). Note that each of these firings is the same as subtracting a column of $\mathcal{L}_1$ from the current configuration.

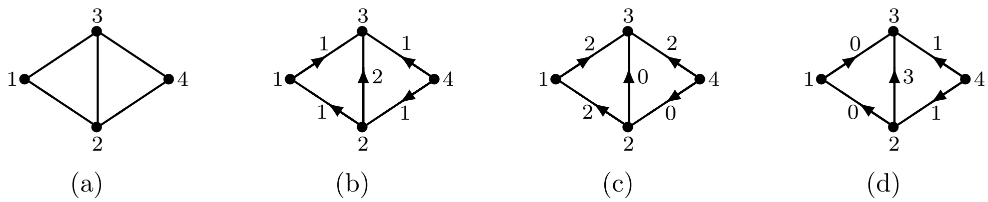

Figure 1.3: Conservative 1-flows and firings

As noted above we have diverted one unit of flow around each of the 2-faces that e_{12}. This firing is precisely the same as subtracting the 12 column of the Laplacian from the configuration.

Let us again return to the the abelian sandpile model. For sandpile graphs, having chosen a sink, we were able to identify critical elements that formed a set of coset representatives for the critical group. Our notion of a critical configuration was predicated on the notion of stability. We defined a configuration stable if $c_i < \deg(v_i)$ for each vertex in G. This worked because firing a vertex could only ever increase or leave unchanged the amount of sand on other vertices.

In higher dimension however, as we saw in the previous example, firing an i-face can actually cause the flow along some other i-faces to decrease. It would perhaps seem natural to then define stability in such a way that we would say an i-face is stable if firing it would force the flow along any i-face into being negative.

Example 1.2.17. Let Σ be the simplicial complex from Figure 1.1. Let us examine the 2$-$skeleton of Σ. A simplicial spanning tree is given by $\{f_{14}, f_{24}, f_{34}\}$ so it suffices to look at configurations on the edges . Suppose we are given the configuration $(2, 2, 1)$ then we have the following

$$(2,2,1) \xrightarrow{e_{12}} (0,3,0) \xrightarrow{e_{13}} (1,1,1)$$
$$(2,2,1) \xrightarrow{e_{13}} (3,0,2) \xrightarrow{e_{23}} (2,1,0).$$

Now, under our definition of stability, both configurations $(1, 1, 1)$ and $(2, 1, 0)$ are stable because no edge can be fired without forcing the flow along another edge to be negative. It follows that this definition of stability fails to force configurations to stabilize uniquely.

It remains an open question of how to naturally define stability in a way that gives a canonical set of representatives analogous to the critical configurations found for sandpile groups.

1.2.4 Relation to Homology Groups

From the previous definitions, it is clear that homology groups and critical groups have similar forms—the homology group is formed by modding out by a few more things than the critical group. It is natural to wonder if there is an easy way to relate these objects. In order to sufficiently answer this question we need to introduce a little bit more machinery. These results come from [2] and a more complete exposition along with proofs can be found there. We will use the concept of simplicial cuts and flows to relate critical groups and homology groups.

Definition 1.2.18. Let Σ be a d-dimensional simplicial complex with associated boundary maps ∂_i. A *cut* of Σ is the set of nonzero faces for an element of $\operatorname{im} \partial_d^*$. A *flow* is an element of $\ker \partial_d$.

Remark 1.2.19. We have unfortunately now overloaded the term flow. In the previous section we used the term flow to literally mean flow across an i-face and in this section we mean the definition given above. Both of these names are consistent with current literature so this overloading is unavoidable. When the word flow appears its meaning should be clear from context.

Example 1.2.20. Let G be the graph in Figure 1.4. By Definition 1.2.18 , we can get a cut for each vertex v by looking at each of the edges in $\partial_1^*(v)$. For instance, a cut about 0 is the edge set $\{01, 12\}$.

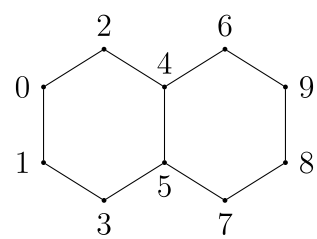

Figure 1.4: Graph for Example 1.2.20

A flow for a graph is simply a cycle since the boundary of a cycle will be zero. For G, the edge sets $\{01, 13, 35, 54, 42, 20\}$ and $\{01, 13, 35, 57, 78, 89, 96, 64, 42, 20\}$ give flows. We can check this by applying the boundary map:

$$\partial_1(12+24+46+65+53+31) = v_1 - v_2 + v_2 - v_4 + v_4 - v_6 + v_6 - v_5 + v_5 - v_3 + v_3 - v_1 = 0.$$

Example 1.2.21. Let Σ be the simplicial complex in Figure 1.5. We can get a cut by looking at $\partial_2^*(\sigma)$ for $\sigma \in F_1(\Sigma)$. For instance $\partial_2(13)$ takes nonzero values for the faces $\{123, 134, 135\}$ which gives a cut.

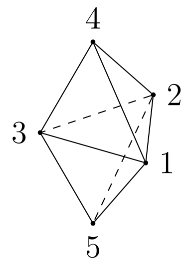

Figure 1.5: Simplicial Complex for Example 1.2.21

The sets of faces $\{123, 124, 134, 234\}$ and $\{124, 125, 134, 135, 234, 235\}$ form flows. Applying ∂ to the first flow (oriented in a cycle) gives us zero:

$$\partial(123-124+134-234) = 12-13+23-12+14-24+13-14+34-23+24-34 = 0.$$

Definition 1.2.22. Let Σ be a pure d-dimensional simplicial complex with associated boundary maps ∂_i. Then we define the *cut lattice* of Σ by

$$\mathcal{C}(\Sigma) = \mathrm{im}_{\mathbb{Z}}\ \partial_d^*,$$

and the *flow lattice* of Σ by

$$\mathcal{F}(\Sigma) = \ker_{\mathbb{Z}} \partial_d.$$

Definition 1.2.23. The *cutflow* group is $\mathbb{Z}^n/(\mathcal{C} \oplus \mathcal{F})$.

Definition 1.2.24. Let A be a group. Then the *torsion summand* is the subgroup of A consisting of element of finite order. We denote this subgroup by

$$\mathbf{T}(A).$$

Theorem 1.2.25. *Let Σ be a pure d-dimensional simplicial complex. Then the following is a short exact sequence*

$$0 \to \mathbb{Z}^n/(\mathcal{C} \oplus \mathcal{F}) \to K_{d-1}(\Sigma) \to \boldsymbol{T}(\overline{H}_{d-1}(\Sigma, \mathbb{Z})) \to 0.$$

Proof. Observe that $\mathrm{im}\,\partial_d\partial_d^* \subseteq \mathrm{im}\,\partial_d \subseteq \ker\partial_{d-1}$. Thus, we can form the short exact sequence

$$0 \to \mathrm{im}\ \partial_d/\mathrm{im}\ \partial_d\partial_d^* \to \ker\partial_{d-1}/\mathrm{im}\,\partial_d\partial_d^* \to \ker\partial_{d-1}/\mathrm{im}\,\partial_d \to 0.$$

The final part of the proof is to show that the cutflow group is isomorphic to $\mathrm{im}\,\partial_d/\mathrm{im}\,\partial_d\partial_d^*$. Applying ∂_d gives

$$\partial_d(\mathbb{Z}^n/(\mathcal{C} \oplus \mathcal{F}) = \mathrm{im}\ \partial_d/\mathrm{im}\ \partial_d\partial_d^*$$

since $\partial_d(\mathbb{Z}^n)$ gives $\mathrm{im}\ \partial_d$ and $\partial_d(\mathrm{im}\ \partial_d^* \oplus \ker\ \partial_d) = \mathrm{im}\ \partial_d\partial_d^*$. Furthermore, ∂_d is injective because $\mathbb{Z}^n/(\mathcal{C} \oplus \mathcal{F}) \subseteq \mathbb{Z}^n/\ker\ \partial_d$ and ∂_d is surjective on $\mathbb{Z}^n/\ker\ \partial_d$. Thus ∂_d gives an isomorphism. This gives the desired short exact sequence. $\square$

Corollary 1.2.26. *Let Σ be a pure d-dimensional simplicial complex. Then the following is a short exact sequence*

$$0 \to \mathbb{Z}^n/(\mathcal{C} \oplus \mathcal{F}) \to K_{i-1}(\Sigma) \to \boldsymbol{T}(\overline{H}_{i-1}(\Sigma, \mathbb{Z})) \to 0.$$

for $1 \leq i \leq d$.

Proof. The i-skeleton of a pure d-dimensional simplicial complex is a pure i-dimensional simplicial complex. By Theorem 1.2.25 the result follows immediately. $\square$

Chapter 2

Morphisms and Products

In this chapter we define a class of mappings that induce homomorphisms on the associated critical groups. We also define two categorical products on abstract simplicial complexes (one for dimensional preserving mappings and one for general mappings)—a tool we will use in chapter three.

2.1 Mappings of Critical Groups

2.1.1 Harmonic Morphisms of Graphs

Baker and Norine introduce a class of graph homomorphisms called harmonic morphisms in [1]. In this section, we will present their definitions and prove that harmonic morphisms induce a group homomorphism between critical groups. In the subsequent section we will generalize these results for simplicial complexes.

Definition 2.1.1. Let G be a graph. We will denote the edge set of G by $E(G)$ and the vertex set by $V(G)$. We say that $x, y \in V$ are *adjacent* if $(x, y) \in E(G)$ and denote this by $x \sim y$.

Definition 2.1.2. Let G and G' be graphs. Let $\phi : G \to G'$. Then ϕ is called a *harmonic morphism* if the following conditions are satisfied:

(1) If $x \sim y$ in $V(G)$, then either $\phi(x) \sim \phi(y)$ or $\phi(x) = \phi(y)$.

(2) For all $x \in V(G)$, $x' \in V(G')$ such that $x' = \phi(x)$ the quantity

$$|\{e \in E(G) : x \in e \text{ and } \phi(e) = e' \ \}|$$

is the same for all $e' \in E(G')$ such that $x' \in e'$.

This definition is actually less complicated then a first read might suggest. Condition (2) states that for for each vertex $\phi(x) = x' \in V(G')$, the number of vertices that are adjacent to x and mapped to x'' is the same for any x'' adjacent to x'. The following examples are offered to help clarify Definition 2.1.2.

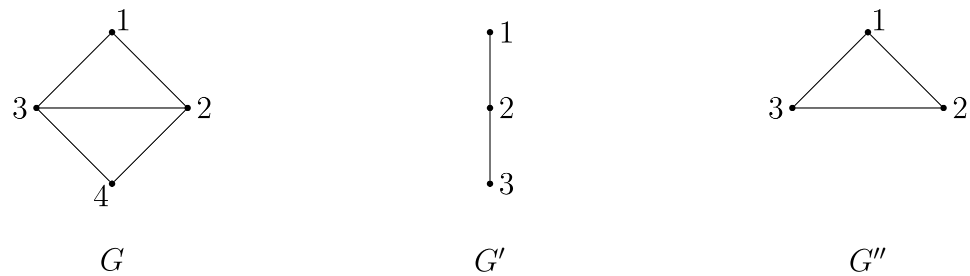

Figure 2.1: Graphs used for Example 2.1.3 and Example 2.1.4.

Example 2.1.3. Take G and G' from Figure 2.1. Let $\phi : G \to G'$ by $\phi(1) = 1$, $\phi(3) = \phi(2) = 2$ and $\phi(4) = 3$. It is easy to check that both conditions (1) and (2) are satisfied by this mapping, thus ϕ is a harmonic morphism.

Example 2.1.4. Take G and G'' from Figure 2.1. Let $\phi : G \to G''$ by $\phi(1) = 1$, $\phi(2) = 2$, $\phi(3) = 3$ and $\phi(4) = 1$. Observe that there are two vertices $\{1, 4\} \in G_1$ that are mapped to $1 \in G_2$ but only one vertex $\{3\} \in G_1$ that is mapped to $3 \in G_2$. Thus ϕ fails condition (2). Thus, ϕ is not a harmonic morphism.

Remark 2.1.5. If ϕ is a harmonic morphism and $e = (x, y) \in E(G)$ then we say that $\phi(e) = \phi(x)$ if $\phi(x) = \phi(y)$ and that $\phi(e) = e' = (\phi(x), \phi(y)) \in E(G')$ if $\phi(x) \neq \phi(y)$.

Before proving the main result of this subsection it is necessary to provide a number of useful facts about harmonic morphisms.

Definition 2.1.6. Let G and G' be simple graphs. Let $\phi : G \to G'$ be a harmonic morphism. Let $x \in V(G)$. Define the *vertical multiplicity* of x as

$$v_\phi(x) := |\{e \in E(G) : x \in e, \phi(e) = \phi(x)\}|$$

and the *horizontal multiplicity* of x as

$$m_\phi(x) := |\{e \in E(G) : x \in e, \phi(e) = e' \in E(G')\}|.$$

Condition (2) of Definition 2.1.2 guarantees that $m_\phi(x)$ is the same no matter which $e' \in E(G')$ is picked. The next lemma gives a relation between the degree of a vertex x and its horizontal and vertical multiplicity.

Lemma 2.1.7. *Let G and G' be simple graphs. Let $\phi : G_1 \to G_2$ be a harmonic morphism.*

$$\deg(x) = \deg(\phi(x))\, m_\phi(x) + v_\phi(x).$$

Proof. The quantity $\deg(x)$ counts the number of vertices adjacent to x in G. The quantity $v_{\phi(x)}$ counts the number of vertices $y \sim x \in V(G)$ such that $\phi(x) = \phi(y)$ and $\deg(\phi(x))\, m_\phi(x)$ counts the number of vertices $y \sim x \in V(G)$ such that $\phi(x) \neq \phi(y)$. The result then follows from condition (1) of Definition 2.1.2. □

Definition 2.1.8. Let $\phi : G \to G'$ be a harmonic morphism. Define the pushforward mapping

$$\begin{aligned}\phi_* : \mathbb{Z}V(G) &\to \mathbb{Z}V(G') \\ c &\mapsto \sum_{v \in V(g)} c_v \phi(x).\end{aligned}$$

Theorem 2.1.9. *Let G and G' be graphs. Let $\phi : G \to G'$ be a harmonic morphism. Then ϕ_* induces a mapping from $K(G)$ to $K(G')$.*

Proof. It suffices to show that ϕ_* maps a firing rule for G to an integer combination of firing rules in G'.

$$\begin{aligned}\phi_* &\left(\deg(x)x - \sum_{(x,y)\in E(G)} y\right) \\ &= \deg(x)\phi(x) - \sum_{(x,y)\in E(G)} \phi(y) \\ &= (\deg(\phi(x))m_{\phi(x)} + v_{\phi(x)})\phi(x) - v_\phi(x)\phi(x) - \sum_{(\phi(x),y')\in E(G')} m_\phi(x)y' \\ &= m_\phi(x)\left(\deg(\phi(x))\phi(x) - \sum_{(\phi(x),y')\in E(G')} y'\right).\end{aligned}$$

Which is an integer multiple of the firing rule for $\phi_*(x)$ in G_2. □

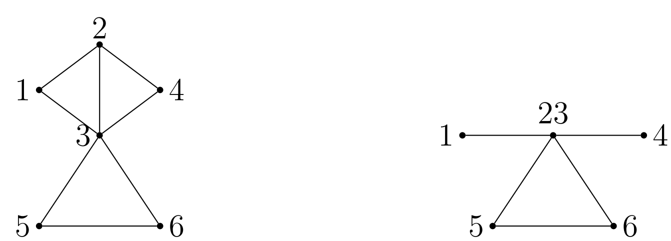

Figure 2.2: Simplicial Complex used for Example

A logical question is whether harmonic morphisms are the only type of morphism that induces a mapping of critical groups. The following example show this is not the case.

Example 2.1.10. Refer to Figure 2.2. Let G be the graph on the left hand side and let G' be the graph on the right hand side. Let $\phi : G \to G'$ be a morphism of graphs defined by $\phi(2) = \phi(3)$ and $\phi(v) = v$ for all other vertices in G. This mapping fails condition (2) for harmonic morphisms. On the other hand, we can check that firing any vertex in G maps to something in the image of the Laplacian of G'. For example, suppose we fire vertex 2 in G. Then we have that

$$\phi_* \begin{pmatrix} -1 \\ 3 \\ -1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} -1 \\ 2 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

The Laplacian for G' is

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{pmatrix}.$$

Then adding columns two, four, and five shows that the firing rule in G is mapped to an integer combination of firing rules in G'.

Theorem 2.1.11. *Let G be a connected graph such that $G = G_1 \cup \cdots \cup G_n$ and $G_i \cap G_j = v \in V(G)$ for all $i \neq j \in [n]$. Then*

$$K_0(G) = K_0(G_1) \times \cdots \times K_0(G_n)$$

Proof. Choose v be the sink vertex and let $A_i = \tilde{\mathcal{L}}$ be the reduced Laplacian for G_i. Then the reduced Laplacian for G is given by

$$\tilde{\mathcal{L}} = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{pmatrix}.$$

Then

$$S(G) = \mathbb{Z}\tilde{V}/\text{im } \tilde{\mathcal{L}} = S(G_1) \times \cdots \times S(G_n).$$

By Theorem 1.2.12, $S(G)$ is isomorphic to $K_0(G)$ and $S(G_i)$ is isomorphic to $K_0(G_i)$ for each i in $[n]$. This gives that

$$K_0(G) = K_0(G_1) \times \cdots \times K_0(G_n).$$

$\square$

Corollary 2.1.12. *Let H be a graph and let G be as in the previous theorem and let $\phi : H \to G$ be a harmonic morphism. Let $U_i = \phi^{-1}(G_i)$. Then if $\phi|_{U_i} : H \to G_i$ is harmonic for each $i \in [n]$, ϕ_* gives a mapping of critical groups.*

Thus, in the previous example, even though ϕ is not technically harmonic, it still induces mappings of critical groups by being harmonic on the subgraphs formed around vertex (23).

2.1.2 Simplicial Morphisms

Our next goal is to find conditions that ensure a morphism of simplicial complexes induces a mapping on the critical groups.

Definition 2.1.13. Let Σ and Σ' be simplicial complexes. Let $\phi : \Sigma \to \Sigma'$. Then ϕ is called a *simplicial morphism* if $\phi(\sigma) \in \Sigma'$ for each $\sigma \in \Sigma$. We will say that a simplicial morphism is *dimension preserving* if for each $\sigma \in \Sigma$, we have $\dim(\sigma) = \dim(\phi(\sigma))$.

Definition 2.1.14. We say that $\sigma, \sigma' \in F_n(\Sigma)$ are *adjacent* if there exists some $\tau \in F_{n+1}(\Sigma)$ such that $\sigma, \sigma' \in \tau$. We denote adjacent faces σ and σ' by $\sigma \sim \sigma$.

The next definition is a generalization of the notion of a harmonic morphism.

Definition 2.1.15. Let Σ be a d-dimensional simplicial complex and let Σ' be a simplicial complex. Let $\phi : \Sigma \to \Sigma'$ be a simplicial morphism. If for all $\sigma \in F_{i-1}(\Sigma)$ the quantity

$$|\{\rho \in F_{i-1}(\Sigma) : \rho \sim \sigma \text{ and } \phi(\rho) = \rho'\}|$$

is the same for all $\rho' \in F_{i-1}(\Sigma')$ such that $\rho' \sim \sigma' = \phi(\sigma)$ then we say ϕ is i-harmonic. A simplicial morphism ϕ that is also i-harmonic will be denoted ϕ_i.

Definition 2.1.16. Let Σ be a d-dimensional simplicial complex with associated boundary maps ∂_i. Let $\tau \in F_{i+1}$ and let $\sigma \in \tau$. Then we say that τ induces a *positive orientation* on σ if $\partial_{i+1}\tau$ has a positive sign for σ and induces a *negative orientation* on σ if $\partial_{i+1}\tau$ has a negative sign for σ.

Example 2.1.17. Let $\Sigma = \{123\}$ be a simplicial complex. Since

$$\partial_2\{123\} = e_{12} - e_{13} + e_{23},$$

the face $\{123\}$ induces a positive orientation on the edges e_{12} and e_{23} and a negative orientation on the edge e_{13}.

We will use the convention that if the orientation induced by σ and the orientation induced by $\phi(\sigma)$ are opposite then we will write $\phi(\sigma) = -\sigma' \in F_n(\Sigma')$. This notion is consistent with our idea of critical groups as a model of discrete flow.

Definition 2.1.18. Let Σ be a simplicial complex and let $\sigma \sim \sigma' \in F_n(\Sigma)$. Then we define

$$\operatorname{sign}(\sigma, \sigma') = \begin{cases} -1 & : \text{if } \sigma \text{ and } \sigma' \text{ induce opposite orientations on their intersection.} \\ 1 & : \text{if } \sigma \text{ and } \sigma' \text{ induce the same orientations on their intersection.} \end{cases}$$

By convention we say that $\operatorname{sign}(\sigma, \sigma) = 1$ and that if σ is not adjacent to σ' then $\operatorname{sign}(\sigma, \sigma') = 0$.

Lemma 2.1.19. *Let $\phi : \Sigma \to \Sigma'$ be a harmonic morphism. Let $\sigma, \sigma' \in F_n(\Sigma)$. Then if (i) $\dim(\sigma) = \dim(\phi(\sigma))$, (ii) $\dim(\sigma') = \dim(\phi(\sigma'))$, and (iii) $\sigma \sim \sigma'$,*

$$\mathit{sign}(\sigma, \sigma') = \mathit{sign}(\phi(\sigma), \phi(\sigma')).$$

Proof. Without loss of generality assume that $\mathrm{sign}(\sigma, \sigma') = 1$. The proof is easy to check by cases

$$\begin{aligned}
\phi(\sigma) = \rho, \phi(\sigma') = \rho' &\Rightarrow \mathrm{sign}(\rho, \rho') = 1 &&\Rightarrow \mathrm{sign}(\phi(\sigma), \phi(\sigma')) = 1\\
\phi(\sigma) = -\rho, \phi(\sigma') = \rho' &\Rightarrow \mathrm{sign}(\rho, \rho') = -1 &&\Rightarrow \mathrm{sign}(\phi(\sigma), \phi(\sigma')) = 1\\
\phi(\sigma) = \rho, \phi(\sigma') = -\rho' &\Rightarrow \mathrm{sign}(\rho, \rho') = -1 &&\Rightarrow \mathrm{sign}(\phi(\sigma), \phi(\sigma')) = 1\\
\phi(\sigma) = -\rho, \phi(\sigma') = -\rho' &\Rightarrow \mathrm{sign}(\rho, \rho') = 1 &&\Rightarrow \mathrm{sign}(\phi(\sigma), \phi(\sigma')) = 1.
\end{aligned}$$

□

We next generalize the concept of horizontal and vertical multiplicity defined above for graphs.

Definition 2.1.20. Let $\phi : \Sigma \to \Sigma'$ be a simplicial morphism and let $\sigma \in F_{n-1}(\Sigma)$. Define the *vertical multiplicity* of ϕ at σ by

$$v_\phi(\sigma) = |\{\tau \in F_n(\Sigma) : \sigma \in \tau \text{ and } \dim(\phi(\tau)) < \dim(\tau)\}|.$$

If ϕ is d-harmonic, then we define the *horizontal multiplicity* of ϕ at σ by

$$m_\phi(\sigma) = |\{\tau \in F_n : \sigma \in \tau \text{ and } \phi(\tau) = \pm\tau'\}|.$$

Since ϕ is d-harmonic, $m_\phi(x)$ is independent of the choice of τ'.

2.1.3 Properties of Simplicial Harmonic Morphisms

Our main goal is to show that simplicial harmonic morphisms induce mappings on critical groups.

Definition 2.1.21. Let $\sigma \in F_n(\Sigma)$ be a simplex. Then we define the k-degree of σ for $k \geq n$ by

$$\deg_k(\sigma) = |\{\tau \in F_{k+1}(\sigma) : \sigma \in \tau\}|.$$

Lemma 2.1.22. *Let $\phi : \Sigma \to \Sigma'$ be a harmonic morphism. Let $\sigma \in F_n(\Sigma)$ be a simplex. Then*

$$\deg_n(\sigma) = \deg_n(\phi(\sigma))\, m_\phi(\sigma) + v_\phi(\sigma).$$

Proof. Let $S = \{\tau \in F_{n+1} : \sigma \in \tau\}$. Clearly $|S| = \deg_n(\sigma)$. Then $\deg_n(\phi(\sigma))\, m_\phi(\sigma)$ is the number of $\tau \in S$ such that $\dim(\tau) > \dim(\phi(\tau))$ and $v_\phi(\sigma)$ is the number of $\tau \in S$ such that $\dim(\tau) = \dim(\phi(\tau))$. The result follows. □

Definition 2.1.23. Let $\phi_n : \Sigma \to \Sigma'$ be a simplicial harmonic morphism. Define the pushforward mapping

$$\begin{aligned}
\phi_* : \mathbb{Z}F_n(\Sigma) &\to \mathbb{Z}F_n(\Sigma')\\
c &\mapsto \sum_{\sigma \in F_n(\sigma)} c_\sigma\, \phi(\sigma).
\end{aligned}$$

Theorem 2.1.24. *Let $\phi_n : \Sigma \to \Sigma'$ be a simplicial morphism. Then ϕ_* induces a mapping from $K_{n-1}(\Sigma)$ to $K_{n-1}(\Sigma')$.*

Proof. It suffices to show that ϕ_* maps a firing rule in Σ to a integer linear combination of firings in Σ'.

$$\begin{aligned}
\phi_*\left(\deg_n(\sigma) - \sum_{\sigma\sim\sigma'} \operatorname{sign}(\sigma,\sigma')\sigma'\right)& \\
&= \deg_n \phi(\sigma) - \sum_{\sigma\sim\sigma'} \operatorname{sign}(\sigma,\sigma')\sigma' \\
&= \deg_n(\sigma)\phi(\sigma) - \sum_{\substack{\sigma\sim\sigma \\ \phi(\sigma)=\phi(\sigma')}} \operatorname{sign}(\sigma,\sigma')\phi(\sigma) - \sum_{\substack{\sigma\sim\sigma' \\ \phi(\sigma)\neq\phi(\sigma')}} \operatorname{sign}(\sigma,\sigma')\phi(\sigma') \\
&= \deg_n(\sigma)\phi(\sigma) - v_\phi(\sigma)\phi(\sigma) - \sum_{\substack{\sigma\sim\sigma' \\ \phi(\sigma)\neq\phi(\sigma')}} \operatorname{sign}(\phi(\sigma),\phi(\sigma'))\phi(\sigma') \\
&= m_\phi(\sigma)\left(\phi(\sigma) - \sum_{\phi(\sigma)\sim\phi(\sigma')} \operatorname{sign}(\phi(\sigma),\phi(\sigma'))\phi(\sigma')\right).
\end{aligned}$$

Which is an integer multiple of firing ϕ_σ in Σ' as desired. □

2.2 Categorical Product of Simplicial Complexes

We will define two categorical products for simplicial complexes: one for degree preserving simplicial morphisms and one for arbitrary simplicial morphisms.

2.2.1 Categorical Products

Definition 2.2.1. Let C be a category and let X and X' be objects in that category. We say that Y is the product of X and X' and denote this by $Y = X \times X'$ if and only if it satisfies the universal property: there exist projection mappings $\pi_X : X \times X' \to X$ and $\pi_{X'} : X \times X' \to X'$ such that for every object Z in C, given morphisms $f : Z \to X$ and $g : Z \to X'$ there exists a unique morphism h such that the diagram in Figure 2.3 commutes.

Example 2.2.2. Let C be the category of sets. Let X and X' be sets, and let $\times$ be the Cartesian product. Then $X \times X' = \{(x, x') : x \in X, x' \in X'\}$. To see that this is the categorical product define the projection mappings in the natural way: $\pi_X(x, x') = x$ and $\pi_{X'}(x, x') = x'$. Let Z be a set and define mappings $f : Z \to X$ and $g : Z \to X'$. In order for the diagram shown in Figure 2.3 to commute, we must have that $h(z \in Z) = (f(z), g(z))$. It follows that the Cartesian product is the categorical product for sets.

$$\begin{tikzcd}
& Z \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] \arrow[d, dotted, "\exists! h"] & \\
& X \times X' \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_{X'}"] & \\
X & & X'
\end{tikzcd}$$

Figure 2.3: Diagram for Categorical Product

2.2.2 Product on Simplicial Complexes - General Simplicial Morphisms

Definition 2.2.3. Let Σ and Σ' be simplicial complexes. Then the product $\Sigma \times \Sigma'$ is a simplicial complex such that

- $F_0(\Sigma \times \Sigma')$ is the Cartesian product $F_0(\Sigma) \times F_0(\Sigma')$
- $\sigma \in F_k(\Sigma \times \Sigma')$ if and only if $\pi_\Sigma(\sigma) \in F_k(\Sigma)$ and $\pi_{\Sigma'}(\sigma) \in F_k(\Sigma')$,

where for $\sigma = ((v_1, v_1'), \dots (v_n, v_n')) \in \Sigma \times \Sigma'$,

$$\begin{aligned} \pi_\Sigma : \Sigma \times \Sigma' &\to \Sigma \\ \sigma &\to (v_1, \dots, v_n) \end{aligned} \qquad\qquad \begin{aligned} \pi_{\Sigma'} : \Sigma \times \Sigma' &\to \Sigma' \\ \sigma &\to (v_1', \dots, v_n'). \end{aligned}$$

Theorem 2.2.4. *The product $\times$ defined in Definition 2.2.3 is the categorical product for abstract simplicial complexes with simplicial morphisms.*

Proof. It is clear that π_Σ and $\pi_{\Sigma'}$ are simplicial morphisms. Thus, it suffices to check that the mapping $h : S \to \Sigma \times \Sigma'$ induced by $f : S \to \Sigma$ and $g : S \to \Sigma'$ is a unique simplicial morphism. Let $v \in F_0(S)$, then the only way for the diagram to commute is to define $h(v) = (f(v), g(v))$. Interpolating faces based on the mapping of vertices then gives a unique simplicial morphism. $\square$

Theorem 2.2.5. *Let Σ, Σ' be simplicial complexes. Then the projection mappings π_Σ and $\pi_{\Sigma'}$ are harmonic.*

Proof. Let $\sigma \in F_k(\Sigma \times \Sigma')$ such that $\pi(\sigma) = \sigma' \in F_k(\Sigma)$. We can express σ' as the vertex set $(v_1, \dots, v_{k+1})$. Let $T = \{\tau \in F_{k+1}(\Sigma \times \Sigma') : \sigma \in \tau\}$ and $T' = \{\tau' \in F_{k+1}(\Sigma) : \sigma' \in \tau'\}$. It suffices to show that for any $\rho, \rho' \in T'$ the same number of things from T are mapped to each. Since $\sigma \in \rho$ and $\sigma' \in \rho$, we can write $\rho = (v_1, \dots, v_{k+1}, v)$ and $\rho' = (v_1, \dots, v_{k+1}, v')$ for some $v, v' \in F_0(\Sigma)$. Let $\nu \in T$ such that $\pi_\Sigma(\nu) = \rho$. Then we can express ν as $((v_1, u_1), \dots, (v_{k+1}, u_{k+1}), (v, u_{k+2}))$ for some $u_1, \dots, u_{k+2}$ in $F_0(\Sigma')$. Then $\nu' = ((v_1, u_1), \dots, (v_{k+1}, u_{k+1}), (v', u_{k+2}))$ is also in T. Then since $\pi(\nu') = T'$ it follows that the same number of faces in T are mapped to each face in T'. $\square$

Corollary 2.2.6. *The projection mappings π_Σ and $\pi_{\Sigma'}$ induce mappings of critical groups.*

Example 2.2.7. Let Σ be a 2-simplex $\{1,2,3\}$. Then the one-skeleton of $\Sigma \times \Sigma$ can be pictured in Figure 2.4.

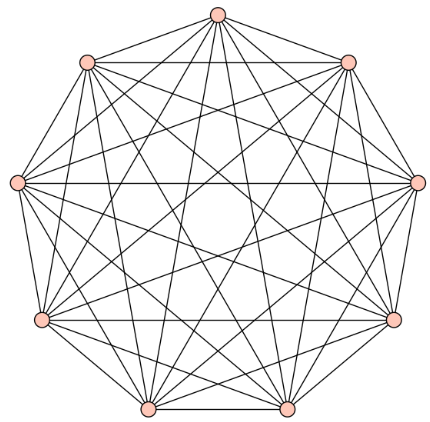

Figure 2.4: 2-simplex $\times$ 2-simplex

2.2.3 Product on Simplicial Complexes - Dimension-Preserving Morphisms

Definition 2.2.8. Let Σ and Σ' be simplicial complexes in the category of simplicial complexes with dimension-preserving morphisms. Then the product $\Sigma \times \Sigma'$ is a simplicial complex such that

- $F_0(\Sigma \times \Sigma')$ is the Cartesian product $F_0(\Sigma) \times F_0(\Sigma')$
- $\sigma \in F_k(\Sigma \times \Sigma')$ if and only if $\pi_\Sigma(\sigma) \in F_k(\Sigma)$ and $\pi_{\Sigma'}(\sigma) \in F_k(\Sigma')$ and $\dim(\sigma)$ $= \dim(\pi_\Sigma(\sigma)) = \dim(\pi_{\Sigma'}(\sigma))$,

where for $\sigma = ((v_1, v_1'), \ldots (v_n, v_n')) \in \Sigma \times \Sigma'$,

$$\begin{aligned} \pi_\Sigma : \Sigma \times \Sigma' &\to \Sigma & \qquad \pi_{\Sigma'} : \Sigma \times \Sigma' &\to \Sigma' \\ \sigma &\to (v_1, \ldots, v_n) & \sigma &\to (v_1', \ldots, v_n'). \end{aligned}$$

Theorem 2.2.9. *The product $\times$ defined in Definition 2.2.8 is the categorical product for abstract simplicial complexes with degree preserving morphisms.*

Proof. The same proof as in 2.2.4 suffices. $\square$

Theorem 2.2.10. *Let Σ, Σ' be simplicial complexes. Then the projection mappings π_Σ and $\pi_{\Sigma'}$ are harmonic.*

$$\begin{tikzcd}
& S \arrow[ddl, bend right, "f"'] \arrow[d, dotted, "\exists ! h"] \arrow[ddr, bend left, "g"] & \\
& \Sigma \times \Sigma' \arrow[dl, "\pi_\Sigma"'] \arrow[dr, "\pi_{\Sigma'}"] & \\
\Sigma & & \Sigma'
\end{tikzcd}$$

Figure 2.5: Diagram for Categorical Product of Simplicial Complexes

Proof. The same proof as in 2.2.5 suffices. $\square$

Corollary 2.2.11. *The projection mappings π_Σ and $\pi_{\Sigma'}$ induce mappings of critical groups.*

Example 2.2.12. Let Σ be a 2-simplex $\{1, 2, 3\}$. Then $\Sigma \times \Sigma$ in the category with dimension-preserving morphisms is pictured in Figure 2.6.

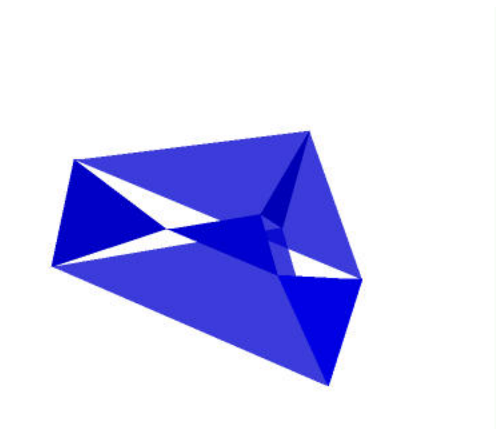

Figure 2.6: 2-simplex $\times$ 2-simplex

Chapter 3

Simplicial Complexes as Discrete Varieties

Let Σ be a d-dimensional simplicial complex. In this section we are going to imagine Σ as analogous to a d-dimensional variety where divisors are formal sums of codimension-1 faces. In this analogy, i-dimensional algebraic cycles corresponds to simplicial i-chains. The critical group $K_i(\Sigma)$ is then closed i-chains modulo conservative flow. Our goal is to show that this is similar to the Chow group of algebraic cycles modulo rational equivalence.

The main objective would be to ultimately form a graded ring

$$R(\Sigma) = \bigoplus_i K_i(\Sigma)$$

whose multiplication encodes simplicial intersection theory.

3.1 Algebraic Geometry

Here we summarize some of the definitions in results of [4] for the purposes of drawing parallels to our own constructions on simplicial complexes.

Definition 3.1.1. Let X be a variety. A k-*cycle*, α, on X is a finite formal sum

$$\alpha = \sum_i n_i V_i,$$

where the V_i are k-dimensional subvarieties of X and the $n_i \in \mathbb{Z}$. The group of k-cycles of X is denoted $Z_k(X)$.

Definition 3.1.2. Let X be a variety. The *field of rational functions* on a variety X is denoted $R(X)$. The nonzero elements form the multiplicative group $R(X)^*$.

Definition 3.1.3. Let X be a variety and let W be a $k+1$ dimensional subvariety. Let r be a rational function on W then define

$$[\mathrm{div}(r)] = \sum \mathrm{ord}_V(r)[V],$$

where we are summing over all codimension one subvarieties V of W.

Definition 3.1.4. A k-cycle, α, is *rationally equivalent to zero*, if there exists a finite number of $k+1$ dimensional subvarieties W_i of X, and $r_i \in R(W_i)^*$, such that

$$\alpha = \sum [\operatorname{div}(r_i)].$$

The cycles rationally equivalent to zero form a subgroup of $Z_k(X)$ denoted $\mathrm{Rat}_k(X)$.

Definition 3.1.5. The group of *k-cycles modulo rational equivalence* on X is the factor group

$$A_k X = Z_k X / \mathrm{Rat}_k(X).$$

Definition 3.1.6. Let $f : X \to Y$ be a proper morphism. Let V be a subvariety of X and define $f(V) = W$. Since f is proper W is a closed subvariety of Y. There is an induced imbedding of $R(W)$ in $R(V)$ which is a finite field extension if W has the same dimension as V. Define

$$\deg(V/W) = \begin{cases} [R(V) : R(W)] & \text{if } \dim(W) = \dim(V) \\ 0 & \text{if } \dim(W) < \dim(V). \end{cases}$$

Definition 3.1.7. Let $f : X \to Y$ be a proper morphism. Then define the pushforward homomorphism

$$f_*[V] = \deg(V/W)[W].$$

This extends to a homomorphism

$$f_* : Z_k X \to Z_k Y.$$

Proposition 3.1.8. *[4] Let $f : X \to Y$ be a proper, surjective morphism of varieties and let $r \in R(X)^*$. Then*

$$f_*[\operatorname{div}(r)] = \begin{cases} [\operatorname{div}(N(r))] & \textit{if } \dim(X) = \dim(Y) \\ 0 & \textit{if } \dim(Y) < \dim(X). \end{cases}$$

where $N(r)$ is the norm of r.

Theorem 3.1.9. *[4] Let $f : X \to Y$ be a proper morphism. Let α be a k-cycle on X such that α is rationally equivalent to zero. Then $f_*\alpha$ is rationally equivalent to zero on Y.*

Proof. The result follows from the previous proposition. □

It follows that f also induces a homomorphism

$$f_* : A_k X \to A_k Y.$$

Definition 3.1.10. Let $f : X \to Y$ be a flat morphism of relative dimension n. Then

$$f^*[V] = [f^{-1}(V)].$$

This extends to give us a pullback homomorphism

$$f^* : Z_k Y \to Z_{k+n} X.$$

Theorem 3.1.11. *[4] Let $f : X \to Y$ be a flat morphism of relative dimension n, and α a k-cycle rationally equivalent to zero on Y. Then $f^*\alpha$ is rationally equivalent to zero in $Z_{k+n}X$.*

Which means that f induces a homomorphism

$$f^* : A_k Y \to A_{k+n} X.$$

3.2 Induced Mappings on Class Groups

For this section, let Σ be a pure d-dimensional simplicial complex.

Definition 3.2.1. We denote by $\mathrm{Div}(\Sigma)$ the free abelian group on $F_{d-1}(\Sigma)$. We can write each element $D \in \mathrm{Div}(\Sigma)$ as $\sum_{\sigma \in F_{d-1}} D(\sigma)(\sigma)$ where $D(\sigma) \in \mathbb{Z}$. We refer to $D \in \mathrm{Div}(\Sigma)$ as a *divisor* on Σ. We denote by $\mathrm{Div}^0(\Sigma)$ the subgroup

$$\{D : D \in \mathrm{Div}(\Sigma)\ ,\ \partial D = 0\ \}\ .$$

Definition 3.2.2. Let $C^k(\Sigma, \mathbb{Z})$ be the group of $\mathbb{Z}$-valued functions on $F_k(\Sigma)$. For $f \in C^k(\Sigma, \mathbb{Z})$ we define the divisor of f by the formulas

$$\mathrm{div}(f) = \sum_{\sigma \in F_k(\Sigma)} \sum_{\substack{\tau \in F_{k+1}(\Sigma) \\ \sigma \in \tau}} \left(f(\sigma) - \sum_{\substack{\sigma \sim \sigma' \\ \sigma' \in \tau}} \mathrm{sign}(\sigma, \sigma') f(\sigma') \right) (\sigma).$$

Divisors of the form $\mathrm{div}(f)$ for some $f \in C^k(\Sigma, \mathbb{Z})$ are called *principal* and the group of principal divisors is denoted $\mathrm{Prin}(\Sigma)$. Clearly $\mathrm{Prin}(\Sigma)$ is a subgroup of $\mathrm{Div}^0(\Sigma)$.

The set of principal divisors is analogous to the Laplacian defined earlier. This can be seen easily by observing that by setting $f(\sigma) = 1$ for some $\sigma \in F_k(\Sigma)$ and zero for all other faces, we recover the firing rule for sigma. We now come to another interpretation of the critical group, this time in the language of algebraic geometry.

The next definition is another framing of the critical group, this time in the language of algebraic geometry.

Definition 3.2.3. The *class group* of Σ, denoted $\mathrm{Cl}(\Sigma)$ is the quotient group

$$\mathrm{Cl}(\Sigma) := \mathrm{Div}^0(\Sigma)/\mathrm{Prin}(\Sigma).$$

Our task is to now define pushforward and pullback mappings that induce mappings on the class groups.

Lemma 3.2.4. *The quantity $|\{\tau \in F_{n+1}(\Sigma) : \phi(\tau) = \tau'\}|$ is independent of the choice of $\tau' \in F_{n_1}(\Sigma')$.*

Proof. Let $\sigma \in F_n(\Sigma')$ and suppose there are simplices $\tau_1, \tau_2 \in F_{n+1}(\Sigma')$ incident to y. Since ϕ is harmonic, for each $\sigma \in F_n(\Sigma)$ such that $\phi(\sigma) = \sigma'$ we have that

$$|\{\tau \in F_{n+1}(\Sigma) : \sigma \in \tau, \phi(\tau) = \tau_1\}| = |\{\tau' \in F_{n+1}(\Sigma) : \sigma \in \tau', \phi(\tau') = \tau_2\}|.$$

Thus,

$$\begin{aligned} |\{\tau \in F_{n+1}(\Sigma) : \phi(\tau) = \tau_1\}| &= \sum_{\sigma \in \phi^{-1}(\sigma')} |\{\tau \in F_{n+1}(\Sigma) : \sigma \in \tau, \phi(\tau) = \tau_1\}| \\ &= \sum_{\sigma \in \phi^{-1}(\sigma')} |\{\tau' \in F_{n+1}(\Sigma) : \sigma \in \tau', \phi(\tau') = \tau_2\}| \\ &= |\{\tau' \in F_{n+1}(\Sigma) : \phi(\tau') = \tau_2\}|. \end{aligned}$$

Now suppose that τ_1, τ_2 are arbitrary simplices in $F_{n+1}(\Sigma')$. Since Σ is n-connected, the result follows by applying the previous result along consecutive edges along a path containing both τ_1 and τ_2. □

Definition 3.2.5. Let ϕ be an n-simplicial harmonic morphism. Then define the degree of ϕ by the formula

$$\deg(\phi) := |\{\tau \in F_{n+1}(\Sigma) : \phi(\tau) = \tau'\}|.$$

By the previous lemma this is well-defined since it is independent of the choice of τ'.

Definition 3.2.6. Let $\phi : \Sigma \to \Sigma'$ be a d-harmonic simplicial morphism. We define the pushforward homomorphism

$$\phi_*(D) := \sum_{\sigma \in F_{d-1}(\Sigma)} D(\sigma)(\phi(x)).$$

Similarly, we define the pullback homomorphism

$$\phi^*(D') := \sum_{\sigma' \in F_{d-1}(\Sigma')} \sum_{\substack{\sigma \in F_{d-1}(\Sigma) \\ \phi(\sigma) = \sigma'}} m_\phi(\sigma) D'(\sigma')(\sigma).$$

Lemma 3.2.7. *Let $\phi : \Sigma \to \Sigma'$. Let Σ' be a pure d-dimensional simplicial complex. For any simplex $\sigma' \in F_{d-1}(\Sigma)'$*

$$deg(\phi) = \sum_{\substack{\sigma \in F_{d-1}(\Sigma) \\ \phi(\sigma) = \sigma'}} m_\phi(x).$$

Proof. Choose a simplex $\tau' \in F_d(\Sigma')$ with $\sigma' \in \tau'$. Then

$$\begin{aligned} \sum_{\sigma \in \phi^{-1}(\sigma')} m_\phi(x) &= \sum_{\sigma \in \phi^{-1}(\sigma')} \sum_{\substack{\tau \in \phi^{-1}(\tau) \\ \sigma \in \tau}} 1 \\ &= |\phi^{-1}(\tau')| \\ &= \deg(\phi). \end{aligned}$$

□

Lemma 3.2.8. *Let $\phi : \Sigma \to \Sigma'$ be a simplicial harmonic morphism, and let $D' \in Div(\Sigma')$. Then $\phi_*(\phi^*(D')) = \deg(\phi)D'$.*

Proof.

$$\begin{aligned}
\phi_*(\phi^*(D')) &= \phi_*\left(\sum_{\sigma' \in F_n(\Sigma')} \sum_{\substack{\sigma \in F_n(\Sigma)\\ \phi(\sigma)=\sigma'}} m_\phi(\sigma) D'(\sigma')(\sigma)\right)\\
&= \sum_{\sigma' \in F_n(\Sigma')} \sum_{\substack{\sigma \in F_n(\Sigma)\\ \phi(\sigma)=\sigma'}} m_\phi(\sigma) D'(\sigma')\phi(\sigma)\\
&= \sum_{\sigma' \in F_n(\Sigma')} D'(\sigma')(\sigma') \sum_{\substack{\sigma \in F_n(\Sigma)\\ \phi(\sigma)=\sigma'}} m_\phi(\sigma)\\
&= \deg(\phi)D'.
\end{aligned}$$

□

Definition 3.2.9. Let $\phi : \Sigma \to \Sigma'$ be a simplicial harmonic morphism and let $f : F_n(\Sigma) \to \mathbb{Z}$ and $f' : F_n(\Sigma') \to \mathbb{Z}$ be functions. Then define $\phi_* f : F_n(\Sigma') \to \mathbb{Z}$ by

$$\phi_* f(\sigma') := \sum_{\substack{\sigma \in F_n(\Sigma)\\ \phi(\sigma)=\sigma'}} m_\phi(\sigma) f(\sigma)$$

and define $\phi^* g : F_n(\Sigma) \to \mathbb{Z}$ by

$$\phi^* := g \circ \phi.$$

Lemma 3.2.10. *Let $\phi : \Sigma \to \Sigma'$ be a harmonic morphism. If dim(Σ') $<$ dim (Σ) then*

$$\phi_*(\mathrm{div}(f)) = 0.$$

Proof. Let $d = \dim(\Sigma)$. It suffices to show that for each $\tau \in F_d(\Sigma)$ and for each

$$\phi(\sigma) - \sum_{\sigma \sim \sigma'} \mathrm{sign}(\sigma, \sigma')\phi(\sigma') = 0.$$

Fix $\tau = (v_1, \ldots, v_{d+1}) \in F_d(\Sigma)$. Let $\sigma_1 = (v_1, \ldots, v_{k-1}, \hat{v_k}, v_{k+1}, \ldots, v_{d+1})$. Then $\sigma_1 \in \tau$. Suppose that there is some v_i, v_j such that $\phi(v_i) = \phi(v_j)$, then by definition $\phi(\sigma_1) = 0$. If there are no such v_i and v_j, then there exists precisely one v_i such that $v_i = v_k$. Let $\sigma_2 = (v_1, \ldots, v_{i-1}, \hat{v_i}, v_{i+1}, \ldots, v_{d+1})$. Then $\sigma_2 \in \tau$, $\sigma_1 \sim \sigma_2$ and $\phi(\sigma_1) = \phi(\sigma_2)$. It follows from the definition of sign that $\mathrm{sign}(\sigma_1, \sigma_2) = 1$. So then

$$\phi(\sigma_1) - \sum_{\sigma_1 \sim \sigma'} \mathrm{sign}(\sigma_1, \sigma')\phi(\sigma') = \phi(\sigma_1) - \mathrm{sign}(\sigma_1, \sigma_2)\phi(\sigma_2) = 0.$$

□

Remark 3.2.11. We have already proved the first part of the next theorem but we offer a second proof that is more consistent with the language used in this section.

Theorem 3.2.12. *Let $\phi : \Sigma \to \Sigma'$ be a harmonic morphism, let $f : F_n(\Sigma) \to \mathbb{Z}$ and let $f' : F_n(\Sigma') \to \mathbb{Z}$. Then*

$$\phi_*(\operatorname{div}(f)) = \operatorname{div}(\phi_* f)$$

and

$$\phi^*(\operatorname{div}(f')) = \operatorname{div}(\phi^* f').$$

Proof. Let's start by rewriting the definition of $\operatorname{div}(f)$

$$\operatorname{div}(f) = \sum_{\substack{\tau \in F_d(\Sigma) \\ \sigma \in \tau}} \left(\left(f(\sigma) - \sum_{\sigma' \sim \sigma} \operatorname{sign}(\sigma, \sigma') f(\sigma') \right) \left((\sigma) - \sum_{\sigma \sim \sigma'} \operatorname{sign}(\sigma, \sigma')(\sigma') \right) \right).$$

Note that here we have used the fact that $\operatorname{sign}(\sigma, \sigma')\operatorname{sign}(\sigma', \sigma'') = \operatorname{sign}(\sigma, \sigma'')$. Now we can calculate

$$\begin{aligned} &\phi_*(\operatorname{div}(f)) \\ &\quad = \sum_{\substack{\tau \in F_d(\Sigma) \\ \sigma \in \tau}} \left(\left(f(\sigma) - \sum_{\sigma' \sim \sigma} \operatorname{sign}(\sigma, \sigma') f(\sigma') \right) \left(\phi(\sigma) - \sum_{\sigma \sim \sigma'} \operatorname{sign}(\sigma, \sigma')\phi(\sigma') \right) \right) \\ &\quad = \sum_{\substack{\tau \in \phi^{-1}(F_d(\Sigma')) \\ \sigma \in \tau}} \left(\left(f(\sigma) - \sum_{\sigma' \sim \sigma} \operatorname{sign}(\sigma, \sigma') f(\sigma') \right) \left(\phi(\sigma) - \sum_{\sigma \sim \sigma'} \operatorname{sign}(\sigma, \sigma')\phi(\sigma') \right) \right), \end{aligned}$$

where the final equality is given by applying the previous lemma. Applying the definition of $\phi_*(f)$ gives

$$\begin{aligned} &\operatorname{div}(\phi_* f)) \\ &= \sum_{\tau' \in F_d(\Sigma')} \left(\left(\sum_{\substack{\sigma \in F_{d-1}(\sigma) \\ \phi(\sigma) = \sigma_1}} m_\phi(\sigma) f(\sigma) - \sum_{\sigma_2 \sim \sigma_1} \sum_{\substack{\sigma' \in F_{d-1}(\Sigma) \\ \phi(\sigma') = \sigma_2}} \operatorname{sign}(\sigma, \sigma') \;\; m_\phi(\sigma') f(\sigma') \right) \right. \\ &\qquad\qquad \left. \left(\sigma_1 - \sum_{\sigma_2 \sim \sigma_1} \operatorname{sign}(\sigma_1, \sigma_2)\sigma_2 \right) \right). \end{aligned}$$

It suffices to show that for every $\tau' \in F_d(\Sigma')$,

$$\sum_{\substack{\tau\in\phi^{-1}(\tau')\\ \sigma\in\tau}}\left(f(\sigma)-\sum_{\sigma'\sim\sigma}\operatorname{sign}(\sigma,\sigma')f(\sigma')\right)$$
$$=\left(\sum_{\substack{\sigma\in F_{d-1}(\sigma)\\ \phi(\sigma)=\sigma_1}} m_\phi(\sigma)f(\sigma)-\sum_{\sigma_2\sim\sigma_1}\sum_{\substack{\sigma'\in F_{d-1}(\Sigma)\\ \phi(\sigma')=\sigma_2}}\operatorname{sign}(\sigma,\sigma')m_\phi(\sigma')f(\sigma')\right),$$

is the same. Fix τ'. Then the equality follows from the definition of m_ϕ.

We now prove the second part of the theorem. Let $g : F_{n-1}(\Sigma') \to \mathbb{Z}$. Let $D' := \operatorname{div}(f')$. Then for every $\sigma \in \Sigma'$.

$$D'(\sigma)=\deg(\sigma)g(\sigma)-\sum_{\sigma'\sim\sigma} g(\sigma).$$

Applying ϕ^* gives

$$\begin{aligned}(\phi^*\operatorname{div}(g))(\sigma) &= (\phi^*D')(\sigma)\\ &= m_\phi(\sigma)D'(\phi(\sigma))\\ &= m_\phi\deg(x)g(\phi(x))-m_\phi(\sigma)\sum_{\sigma'\sim\sigma} g(\sigma).\end{aligned}$$

Now take,

$$\begin{aligned}\operatorname{div}(\phi^*g)(\sigma) &= \operatorname{div}(g\circ\phi)(\sigma)\\ &= m_\phi(\sigma)\deg(\sigma)g(\phi(\sigma))-\sum_{\sigma\sim\sigma'} g(\phi(\sigma')).\end{aligned}$$

By lemma 2.1.22

$$\deg(\sigma)g(\phi(\sigma)) = m_\phi(\sigma)\deg(\phi(\sigma))g(\phi(\sigma))+\sum_{\substack{\sigma'\sim\sigma\\ \phi(\sigma)'=\phi(\sigma)}} g(\phi(\sigma')).$$

Substituting this gives

$$m_\phi(\sigma)\deg(\sigma)g(\phi(\sigma))-\sum_{\substack{\sigma'\sim\sigma\\ \phi(\sigma)'\neq\phi(\sigma)}} g(\phi(\sigma')).$$

By definition of $m_\phi(x)$ this gives

$$m_\phi(\sigma)\deg(\sigma)g(\phi(\sigma))-m_\phi(\sigma)\sum_{\sigma'\sim\sigma\in F_n(\Sigma')} g(\sigma'),$$

which completes the proof. □

Corollary 3.2.13. *If $\phi : \Sigma \to \Sigma'$ is a simplicial harmonic morphism, then*

$$\phi_*(\operatorname{Prin}(\Sigma)) \subseteq \operatorname{Prin}(\Sigma').$$

and

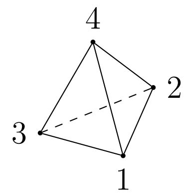

Figure 3.1: Simplicial Complex used for Example 3.3.1

$$\phi_*(\text{Prin}(\Sigma)) \subseteq \text{Prin}(\Sigma').$$

It follows that ϕ_* induces a group homomorphism

$$\phi_* : \text{Cl}(\Sigma) \to \text{Cl}(\Sigma'),$$

and that ϕ^* induces a group homomorphism

$$\phi^* : \text{Cl}(\Sigma') \to \text{Cl}(\Sigma).$$

3.3 The Analogy Breaks Down

Things have looked good so far. We have manged to find a pushforward and a pullback mapping that induce a mapping on class groups. Unfortunately, even attempting to define intersection products for low dimensional structures is difficult as demonstrated by the next example.

Example 3.3.1. Let Σ be the simplicial complex pictured in Figure 3.1. Recall that $K_1(\Sigma)$ is $\mathbb{Z}_4$ and that $K_0(\Sigma) = \mathbb{Z}_4 \times \mathbb{Z}_4$. Form three cycles: $C_1 = e_{12} + e_{23} - e_{13}$, $C_2 = e_{13} + e_{34} - e_{14}$ and $C_3 = e_{12} + e_{24} - e_{14}$. Observe that

$$C_1 - C_2 = 2e_{13} - e_{34} + e_{14} + e_{12} + e_{13},$$

is equivalent to zero. Then $C_1 \sim C_2$, ie C_1 is rationally equivalent to C_2. Suppose that we wish to define an the intersection product $\cap$. We need the following to hold

$$C_3 \cap C_1 \sim C_3 \cap C_2.$$

Ideally, our intersection would be defined only on the actual intersection of the cycles (we want $C_1 \cap C_3$ to be some formal sum of v_1 and v_2). Let's suppose that

$$C_3 \cap C_1 = av_1 - av_2$$

for some $a \in \mathbb{Z}$. Now, there is no $b \in \mathbb{Z}$ such that $\pm(bv_1 - bv_3)$ is equivalent to $av_1 - av_2$. This demonstrates that a nonzero intersection product can't be defined on the intersection of vertices in the cycle. We would also ideally be able to form our ring by taking

$$\mathbb{Z}[X_1, \ldots X_n]/I,$$

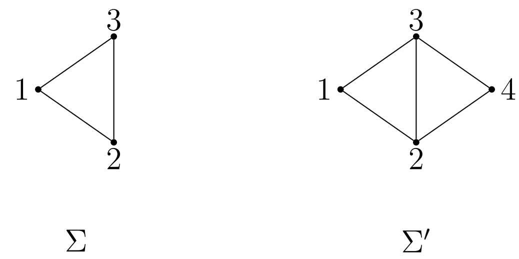

Figure 3.2: Simplicial complexes used for Example 3.3.3.

where I is an ideal that encodes the relations on the generators and their products, and the X_i are generators of $K_1(\Sigma)$. But is $K_1(\Sigma)$ generated by a single element and $K_0(\Sigma)$ requires two generators, so then the ring $K_1(\Sigma) \oplus K_0(\Sigma)$, cannot be described in the form above.

Furthermore, inclusion mappings are not harmonic.

Example 3.3.2. Let Σ, Σ' be the simplicial complexes in Figure 3.2 and let $i : \Sigma \to \Sigma'$ be the inclusion mapping. It is easy to check that $i_*(1, -2, 1) = (1, -2, 1, 0)$ is not in the image of the Laplacian for Σ'.

It is similarly unclear how to define a pullback for inclusion mappings that induces a mapping of critical groups.

References

[1] Matthew Baker and Serguei Norine, *Harmonic morphisms and hyperelliptic graphs*, Int. Math. Res. Not. IMRN (2009), no. 15, 2914–2955.

[2] A. M. Duval, C. J. Klivans, and J. L. Martin, *Cuts and flows of cell complexes*, ArXiv e-prints (2012).

[3] Art M. Duval, Caroline J. Klivans, and Jeremy L. Martin, *Critical groups of simplicial complexes*, Ann. Comb. **17** (2013), no. 1, 53–70.

[4] William Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 2, Springer-Verlag, Berlin, 1984. MR 732620 (85k:14004)

[5] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR 1867354 (2002k:55001)

[6] David Perkinson, Jacob Perlman, and John Wilmes, *Primer for the algebraic geometry of sandpiles*, Conference Proceedings 2011 Bellairs Workshop in Number Theory (To appear).