Invariant and Covariant Rings
of Finite Pseudo-Reflection Groups

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I dedicate my thesis to Neal Stolzfus and *Finite Reflection Groups* by Benson and Grove who introduced me to this subject and to my advisor, David Perkinson, who helped me understand it. Also, special thanks to Tony Geramita and Lorenzo Robbiano for their helpful suggestions.

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Abstract

In this thesis we look at the polynomial invariants and covariants of finite groups generated by pseudo-reflections and calculate the ideal for the orbit of a single point.

Introduction

Many books have been written on the subjects of invariant theory and its relation to pseudo-reflection groups. However, most of them present the material in a very abstract way and consequently at a level a bit beyond most undergraduates. In my thesis I have tried to give an account of the invariant theory of pseudo-reflection groups which is more concrete and down to earth. It is my hope that this thesis can serve as a more accesible introduction to the subject.

In keeping with this goal, the first chapter provides a basic overview to reflection and pseudo-reflection groups while the second chapter gives an introduction to their invariant theory. The main result of the third chapter is a computation of the ideal and Hilbert function of the orbit of a generic point under a pseudo-reflection group. From these calculations, a conjecture is made relating the Hilbert function of the whole group to the Hilbert function of a certain subgroup. The fourth chapter proves some basic facts about the covariant ring and harmonics of a pseudo-reflection group.

Chapter 1

Reflection and Pseudo-Reflection Groups

1.1 Reflection Groups

The mathematical definition of a reflection is made in complete accordance with our nonmathematical notion, as a "flip" through some hyperplane inside $\mathbb{R}^n$.

Definition 1.1.1. A *reflection* is a linear isomorphism $s : \mathbb{R}^n \to \mathbb{R}^n$ which fixes pointwise a hyperplane, H_s, and sends any vector orthogonal to H_s to its negative. H_s is called the hyperplane of s and the orthogonal vector, α_s, is called the *root vector* (or simply the *root*) of s. Note that α_s is only defined up to a constant.

From this definition, we derive the following formula for the action of s on $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$. Then $s(x) = x - \frac{2\langle x,\alpha\rangle}{\langle \alpha,\alpha\rangle}\alpha$. This subtracts from our original vector twice its component in the direction of α, thus reflecting it about the line perpendicular to α. One important property of reflections is that they are orthogonal, or inner product preserving, transformations i.e. $\langle sx, sy\rangle = \langle x, y\rangle$. This can be verified from the formula given above.

One easy way to think about reflections is as $n \times n$ real matrices. Since our space decomposes as $H_s \oplus \mathbb{R}\alpha_s$, we can choose a basis for $\mathbb{R}^n$ by first choosing a basis $x_1, \ldots, x_{n-1}$ for the hyperplane H_s and then completing it to a basis for the whole space by adding the vector $x_n = \alpha_s$. With respect to this basis, s is a diagonal matrix whose diagonal entries are all one except for the last one which is -1. The eigenvalue not equal to one, λ_s, is called the *exceptional eigenvalue* of s. Thus for any reflection, s, we have $\det(s) = \lambda_s = -1$.

Definition 1.1.2. A group $G \subset \mathrm{GL}(\mathbb{R}^n)$ is called a *reflection group* if it is generated by reflections. (Note that in G each reflection s has order 2.)

Example 1.1.3. In $\mathbb{R}^2$ the finite reflection groups are precisely the dihedral groups. [7, pages 12–14]

When studying reflection groups, we have two main ways to extract information: by looking at either roots or hyperplanes.

Definition 1.1.4. A *root system* for a reflection group is a set

$$\Delta = \{\alpha_s \mid s \in G \text{ a reflection}, ||\alpha|| = 1\}.$$

The root system for a reflection group usually has order greater than the dimension of $\mathbb{R}^n$. This means the root vectors $\alpha_s \in \Delta$ are probably not linearly independent. What we would like is some analog of a basis for the set Δ.

Definition 1.1.5. A *fundamental system* for a root system Δ is a subset $\Pi \subset \Delta$ which satisfies the following properties:

1. The elements of Π are linearly independent.
2. For each $\alpha \in \Delta$, $\alpha = \sum_{\beta\in\Pi} \lambda_i \beta_i$ where for all i $\lambda_i \geq 0$ or $\lambda_i \leq 0$.

Given some root system for a reflection group, the easiest way to find a fundamental system is to use our knowledge of the group's hyperplanes.

Definition 1.1.6. A *chamber* of a reflection group G is one connected component of $\mathbb{R}^n - \{H_s \mid s \in G\}$. The hyperplanes which form the boundary of a chamber are called its *walls*.

Using this definition, finding a fundamental system for our group becomes much easier. Given a reflection group G and a root system Δ, a fundamental system, Π, for Δ can be found by choosing any single chamber, C. For each hyperplane, H_s, there are two vectors in Δ which are orthogonal to H_s. A root vector α_s is in Π if its hyperplane H_s is a wall of C and α_s is on the same side of H_s as C.

Example 1.1.7. Let G be the symmetry group of a square in $\mathbb{R}^2$. The hyperplanes of this group are given by the lines $y = x$, $y = -x$, $y = 0$ and $x = 0$. Our root system Δ is the set of vectors $\pm(0,1)$, $\pm(1,0)$, $\pm(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $\pm(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$. If we take as our chosen chamber the set in the first quadrant bounded by the x-axis and the line $y = x$, we get as our fundamental system Π the vectors $\vec{\beta}_1 = (0,1)$ and $\vec{\beta}_2 = (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$.

This idea of a fundamental system turns out to be precisely the one we need to obtain a very simple group presentation for any reflection group G. A theorem of Coxeter states that every reflection group has a presentation of the following form, where Π is a fundamental system of G. Let $S = \{s \in \ G \mid \alpha_s \in \Pi\}$. For every pair of elements, s_i and s_j in S, let m_{ij} be the order of the product $s_i s_j$ in G. Then $G = \langle s \in S \mid (s_i s_j)^{m_{ij}} = 1\rangle$. (Obviously if $i = j$ then m_{ij} is 1.) This amazing result tells us that these simple relations are all that is necessary to completely define our group G.

Using these group presentations, the finite reflection groups have been completely classified as four infinite families and six exceptional groups.

The four infinite families of reflection groups are: A_n (for $n \geq 1$), B_n/C_n (for $n \geq 2$), D_n (for $n \geq 4$), and $I_2(n)$ (for $n \geq 3$). The family B_n/C_n splits into two separate families if we require its root system to form a $\mathbb{Z}$-lattice. (This is used in Lie theory.) Since in this thesis we will never look at lattices formed by root systems, we will ignore the distinction and refer to it simply as B_n.

The first type of group is

$$A_n = \langle s_i \mid (s_i s_j)^a = 1\rangle$$

where $\alpha_{s_i} = e_i - e_{i+1}$ for $i \in \{1, \ldots, n-1\}$ and $a = 3$ if $|i - j| = 1$, $a = 1$ if $i = j$, and $a = 2$ otherwise. A_n is more commonly known as the symmetric group on n variables, and it has root system $\{e_i - e_j\}$ where e_i is the ith standard basis vector.

The second type of group is

$$B_n = \left\langle s_i, t \mid (s_i s_j)^a = 1, (s_i t)^b = 1 \right\rangle$$

where s_i and a are as above, $\alpha_t = e_1$ and $b = 4$ if $i = 1$ and 2 otherwise. B_n is the group of permutations and sign changes on n variables. It has root system $\{\pm e_i \pm e_j, \pm e_i\}$.

The third group type is

$$D_n = \langle s_i, r \mid (s_i s_j)^a = 1, (s_i r)^c = 1\rangle$$

where s_i and a are as above, $\alpha_r = e_1 + e_2$ and $c = 3$ if $i = 2$ and 2 otherwise. D_n is the group of permutations and even numbers of sign changes on n variables. Its root system is $\{\pm e_i \pm e_i\}$.

The final family of groups, $I_2(n)$ is just the set of dihedral groups. Here

$$I_2(n) = \langle e, f \mid (ef)^n = 1\rangle$$

where $\alpha_e = e_1$, α_f is the vector of unit length in the first quadrant and the angle between α_e and α_f is $\frac{2\pi}{n}$.

The exceptional groups are usually called E_6, E_7, E_8, F_4, H_3, and H_4. For the most part, they are hard to visualize, but I will attempt to give some idea of how it can be done.

F_4 is the group of symmetries of the regular solid in $\mathbb{R}^4$ which has 24 octahedral faces.

H_3 is the group of symmetries of the icosahedron (or dodecahedron) in $\mathbb{R}^3$.

H_4 is the symmetry group of the regular solid in $\mathbb{R}^4$ having 120 dodecahedral faces or alternately of the regular solid in $\mathbb{R}^4$ with 600 tetrahedral faces.

For the groups E_6, E_7, and E_8 I will simply give fundamental systems.

Starting with the largest group,

$$E_8 = \left\langle w_i \mid (w_1w_3)^3 = (w_3w_4)^3 = (w_2w_4)^3 = (w_4w_5)^3 = (w_5w_6)^3 = (w_6w_7)^3 = (w_7w_8)^3 = 1\right\rangle$$

where $\alpha_{w_1} = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8)$, $\alpha_{w_2} = e_1 + e_2$, and $\alpha_{w_i} = e_{i-1} - e_{i-2}$ for $i = 3, \ldots, 8$. (Any product w_iw_j not mentioned in the relations is assumed to have order 2.)

Using the same notation,

$$E_7 = \left\langle w_i \mid (w_1w_3)^3 = (w_3w_4)^3 = (w_2w_4)^3 = (w_4w_5)^3 = (w_5w_6)^3 = (w_6w_7)^3 = 1\right\rangle$$

for $i = 2, \ldots, 7$ with $\alpha_{w_1} = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 + e_7)$, and

$$E_6 = \left\langle w_i \mid (w_1w_3)^3 = (w_3w_4)^3 = (w_2w_4)^3 = (w_4w_5)^3 = (w_5w_6)^3 = 1\right\rangle$$

for $i = 2, \ldots, 6$ with $\alpha_{w_1} = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 + e_6)$.

1.2 Pseudo-Reflection Groups

So far, all our results have been strictly over the real numbers. However, since we eventually want to look at polynomial rings related to these groups, we would like to somehow generalize the notion of a reflection group to the complex numbers.

Definition 1.2.1. A map $s : \mathbb{C}^n \to \mathbb{C}^n$ is called a *pseudo-reflection* if it is a non-identity linear isomorphism with finite order which fixes pointwise a hyperplane, H_s.

As for real reflections, it is easy to see that all such s are diagonalizable and that their eigenvalues are all 1 except for the one corresponding to a vector orthogonal to H_s. Since s must have finite order, this exceptional eigenvalue must be a root of unity, $\zeta_m = \mathrm{e}^{\frac{2\pi i}{m}}$ for some $m \in \mathbb{Z}$.

Definition 1.2.2. A group $G \subset \mathrm{GL}(\mathbb{C}^n)$ is a *pseudo-reflection group* if G is generated by pseudo-reflections.

Chapter 2

The Invariant Ring

2.1 Invariant Rings

If we take $\{t_1, \ldots, t_n\}$ to be a basis for $\mathbb{C}^n$, we can dualize to the space of polynomial functions $f : \mathbb{C}^n \to \mathbb{C}$ which can be represented as $\mathbb{C}[x_1, \ldots, x_n]$ where each $x_i \in (\mathbb{C}^n)^*$ is dual to t_i. We can then define the action of a pseudo-reflection, s, on a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ by $sf(\vec{x}) = f(s^{-1}\vec{x})$. (It is necessary to use s^{-1} rather than s in order to preserve associativity.)

Definition 2.1.1. The invariant ring of a group, $G \subset \mathrm{GL}(\mathbb{C}^n)$ is

$$\mathbb{C}[x_1, \ldots, x_n]^G = \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid sf = f \quad \forall s \in G\}.$$

Obviously we would like to know something about the structure of this ring. By a theorem of Hochster and Eagan, for all finite matrix groups $G \subset \mathrm{GL}(\mathbb{C}^n)$ the ring of invariants, $\mathbb{C}[\vec{x}]^G$, will be Cohen-Macaulay. This means that there exist sets of polynomials, $\theta_1, \ldots, \theta_n$, and $\eta_1, \ldots, \eta_m$ such that the θ_i's are algebraically independent and $\mathbb{C}[\vec{x}]^G = \bigoplus_{k=1}^m \eta_k \mathbb{C}[\theta_1, \ldots, \theta_n]$ as vector spaces over $\mathbb{C}$. This presentation of the ring is called the *Hironaka decomposition* and the θ's and η's are called the *primary invariants* and *secondary invariants* respectively. [11, pages 37–40]

We would like to know when we can write the ring simply in terms of the θ_i's, or rather when $m = 1$ and $\eta_1 = 1$. This means that the ring of invariants would be a polynomial algebra. The theorem of Shepard, Todd and Chevalley tells us that a group's ring of invariants is a polynomial algebra if and only if that group is a finite pseudo-reflection group.

It is very important to note that this theorem is true as stated only because we are working over $\mathbb{C}$ and $\mathrm{char}(\mathbb{C}) = 0$. This is not the case if the characteristic of the field over which we are working divides the order of G. In that case, we retain the theorem that if $\mathbb{C}[\vec{x}]^G$ is a polynomial algebra then G is a finite pseudo-reflection group. However we no longer have that G a finite pseudo-reflection group implies that $\mathbb{C}[\vec{x}]^G$ is a polynomial algebra.

Example 2.1.2. To illustrate the idea of invariant rings, here is a list of the three main families of reflection groups and their fundamental invariants.

Probably the best known example of the Shepard-Todd-Chevalley theorem is when $G = A_n$, the symmetric group on n elements. Letting A_n act on the set of polynomials $\mathbb{C}[\vec{x}]$ by permuting the x_i's, we have $\mathbb{C}[\vec{x}]^G = \mathbb{C}[\sigma_1, \ldots, \sigma_n]$ where σ_i is the ith elementary symmetric function.

When $G = B_n$, the group acts on $\mathbb{C}[\vec{x}]$ by permuting the x_i's and changing the signs of an arbitrary number of variables. The ring of invariants for G is
$\mathbb{C}[\vec{x}]^G = \mathbb{C}[\tau_1, \ldots, \tau_n]$ where $\tau_i = \sigma_i(x_1^2, \ldots, x_n^2)$.

When $G = D_n$, the group acts by permuting the x_i's and changing the signs of an even number of variables. The ring of invariants for G is $\mathbb{C}[\vec{x}]^G = \mathbb{C}[\tau_1, \ldots, \tau_{n-1}, \sigma_n]$.

The set $\theta_1, \ldots, \theta_n$ of primary invariants is closely linked to the group G. The proof of the Shepard-Todd-Chevalley Theorem [11, pages 44–49] proceeds by first taking a basis for the ideal generated by the invariants of G. By the Hilbert Basis Theorem, there exists some finite basis for this ideal. This finite list of polynomials is shown to be our algebraically independent list of primary invariants, the θ_i's.

An important step in this proof, the following theorem describes the relationship between G and the primary invariants of its invariant ring. We will need this result in chapters three and four. It is based on the proof [11, pages 47–48]. The proof requires the following definition.

Definition 2.1.3. The *Hilbert series* of a graded ring, $R = \oplus_\ell R_\ell$, is

$$\Phi_R(z) = \sum_{\ell \geq 0} H_R(\ell) z^\ell,$$

where $H_R(\ell) = \dim(R_\ell)$ is the *Hilbert function* of R. If R is finitely generated as a module over $\mathbb{C}[\vec{x}]$, the Hilbert function, $H_R(\ell)$, is equal to a polynomial function for large values of ℓ. This polynomial is called the *Hilbert polynomial*.

Theorem 2.1.4. *Let $G \subset \mathrm{GL}(\mathbb{C}^n)$ be a finite pseudo reflection group with invariant ring $\mathbb{C}[\theta_1, \ldots, \theta_n]$. If we let $\deg(\theta_i) = d_i$ and r equal the number of pseudo-reflections in G, then $|G| = \prod_{i=1}^n d_i$ and $r = \sum_{i=1}^n (d_i - 1)$.*

Proof. This proof will proceed by computing the Laurent series of the Hilbert series of $\mathbb{C}[\vec{x}]^G$ about the point $z = 1$ in two different ways and then equating their coefficients.

By a theorem of Molien, the Hilbert series of the invariant ring $\mathbb{C}[\vec{x}]^G$ of a finite matrix group G is given by

$$\Phi_G(z) = \frac{1}{|G|} \sum_{\pi \in G} \frac{1}{\det(id - z\pi)}.$$

(Here id represents the $n \times n$ identity matrix.)

Looking at the individual terms, we see that

$$\begin{aligned} \det(id - z\pi)^{-1} &= (-1)^n \det(z\pi - id)^{-1} = (-z)^n \det(\pi - \tfrac{1}{z} id)^{-1} \\ &= (-z)^n \prod_{i=1}^n \left(\lambda_i - \frac{1}{z} \right)^{-1} = \prod_{i=1}^n \frac{1}{1 - z\lambda_i} \end{aligned}$$

where the λ_i's are the eigenvalues of π.

This means that the highest-order pole about $z = 1$ has order n and occurs when π has all eigenvalues equal to one. Because all of these matrices are over $\mathbb{C}^n$ with finite order, they are diagonalizable and so only $\pi = id$ has all eigenvalues equal to one.

By similar reasoning, $\prod_{i=1}^n \frac{1}{1 - z\lambda_i}$ has a pole of order $n - 1$ at $z = 1$ whenever $\lambda_i = 1$ for all but a single value of i. This occurs only when π is a pseudo-reflection. Thus the coefficient of $(1 - z)^{-n+1}$ is $\sum_\sigma (1 - \det(\sigma))^{-1}$ summing over all pseudo-reflections, σ, in G since $\det(\sigma) = \lambda$ for the exceptional eigenvalue, $\lambda \neq 1$, of σ.

If σ is a pseudo-reflection, σ^{-1} is also a pseudo-reflection. Thus

$$\begin{aligned} 2 \sum_\sigma \frac{1}{1 - \det(\sigma)} &= \sum_\sigma \left(\frac{1}{1 - \det(\sigma)} + \frac{1}{1 - \det(\sigma^{-1})} \right) \\ &= \sum_\sigma \left(\frac{1}{1 - \det(\sigma)} + \frac{1}{1 - (\det(\sigma))^{-1}} \right) = \sum_\sigma 1 = r. \end{aligned}$$

This means that about $z = 1$, the Laurent series is

$$\Phi_G(z) = \frac{1}{|G|}(1 - z)^{-n} + \frac{r}{2|G|}(1 - z)^{-n+1} + \mathcal{O}((1 - z)^{-n+2}).$$

Since the θ_i's are algebraically independent, a second formulation of the Hilbert series of $\mathbb{C}[\theta_1, \dots, \theta_n]$ is

$$\Phi_G(z) = \prod_i \frac{1}{(1 - z^{d_i})}.$$

The term $(1 - z^{d_i})$ can be rewritten as $(1 - z)(1 + z + \cdots + z^{d_i - 1})$ so this product becomes

$$\prod_i \frac{1}{(1 - z)(1 + \cdots + z^{d_i - 1})} = \frac{1}{(1 - z)^n} \prod_i \frac{1}{1 + \cdots + z^{d_i - 1}}.$$

Expanding this expression as a Laurent series about the point $z = 1$, we see the highest pole is of order n. Multiplying the above expression by $(1 - z)^n$ and taking the limit as z goes to 1, we get the corresponding coefficient in the Laurent series:

$$\prod_i \frac{1}{d_i} = \frac{1}{d_1 \dots d_n}.$$

To find the coefficient for the pole of order $n - 1$ we form a new function $\Phi_G(z) - \prod_{i=1}^n \frac{1}{d_i(1-z)}$ and find the coefficient of its highest order pole. This coefficient equals

$$\begin{aligned}
&\lim_{z \to 1} (1 - z)^{n-1} \left[\Phi_G(z) - \prod_i \frac{1}{d_i(1 - z)} \right] \\
&= \lim_{z \to 1} \frac{1}{1 - z} \left[\prod_i \frac{1}{1 + ... + z^{d_i - 1}} - \prod_i \frac{1}{d_i} \right] \\
&= \lim_{z \to 1} \frac{1}{1 - z} \left[\frac{\prod_i d_i - \prod_i (1 + \cdots + z^{d_i - 1})}{\prod_i d_i \prod_i (1 + \cdots + z^{d_i - 1})} \right] \\
&= \frac{-1}{(\prod_i d_i)^2} \lim_{z \to 1} \frac{\prod_i d_i - \prod_i (1 + \cdots + z^{d_i - 1})}{z - 1}.
\end{aligned}$$

The numerator of this function vanishes at $z = 1$ so, taking its derivative, the first order term at $z = 1$ has coefficient

$$\frac{-1}{(\prod_i d_i)^2} \lim_{z \to 1} - \sum_j \left((1 + 2z + \cdots + (d_j - 1) z^{d_j - 2}) \prod_{i \neq j} (1 + \cdots + z^{d_i - 1}) \right)$$

$$= \frac{1}{(\prod_i d_i)^2} \lim_{z \to 1} \prod_i (1 + \cdots + z^{d_i - 1}) \sum_j \frac{1}{d_j} (1 + 2z + \cdots + (d_j - 1) z^{d_j - 2}).$$

Setting $z = 1$ we see that this is just

$$\frac{\prod_i d_i}{(\prod_i d_i)^2} \sum_i \frac{d_i(d_i - 1)}{2d_i} = \frac{\sum_i (d_i - 1)}{2 \prod_i d_i}.$$

Thus the Laurent series about the point $z = 1$ is

$$\Phi_G(z) = \frac{1}{\prod_i d_i}(1-z)^{-n} + \frac{\sum_i(d_i - 1)}{2\prod_i d_i}(1-z)^{-n+1} + \mathcal{O}((1-z)^{-n+2}).$$

Equating the coefficients of these two Laurent series, we see that $|G| = \prod_i d_i$ and $r = \sum_i(d_i - 1)$. □

2.2 The Invariant Ring of G^1

The invariant ring of a finite pseudo-reflection group, G, is completely understood over $\mathbb{C}$ since $\mathrm{char}(\mathbb{C}) = 0$, so we will turn our attention to one of G's subgroups. For the rest of this section, we will have $G \subset \mathrm{GL}(\mathbb{C}^n)$ be a finite pseudo-reflection group.

Definition 2.2.1. Let $G^1 = \{s \in G \mid \det(s) = 1\} < G$.

Note that this subgroup is normal in G, because if $g \in G^1$, $h \in G$ then $\det(h^{-1}gh) = \det(h^{-1})\det(g)\det(h) = 1$. We can therefore create the factor group G/G^1. The group G/G^1 is cyclic. To see this note that each coset of G/G^1 is a set of elements of G which have the same determinant. The determinant of an element of G is $e^{2\pi i/m}$ for some $m \in \mathbb{Z}$, so we can take the lcm of the m's involved. There exists an element, t, of G which has determinant $e^{2\pi i/lcm(m)}$. Since t's determinant generates all the determinants of elements in G, we have $G/G^1 = \langle t \rangle$.

We are interested in the structure of this group's invariant ring. We know that it is Cohen-Macaulay and therefore can be described completely by two sets of polynomials, the θ_i's and the η_k's. Keeping that in mind, we need the following definitions before we proceed to the main theorem.

Definition 2.2.2. Let $\chi : G \to \mathbb{C}^*$ be a linear homomorphism. We define a polynomial $f \in \mathbb{C}[\vec{x}]$ to be χ-invariant if $\forall s \in G$, $sf = \chi(s)f$. The set of all such functions is denoted $\mathbb{C}[\vec{x}]^G_\chi$. We will define χ_k by $\chi_k(s) = \det^{-k}(s)$ for any $k \in \mathbb{Z}$.

For each pseudo-reflection $s \in G$, there is a hyperplane fixed by s which we call H_s. Let L_{H_s} be the linear function defining the hyperplane H_s.

Definition 2.2.3. Let $\mathcal{H}(G)$ be the set of all hyperplanes fixed by pseudo-reflections in G. Then we define $f_{\chi_k} = \prod_{H \in \mathcal{H}(G)}(L_H)^k$.

The first thing to note is that f_{χ_k} is χ_k-invariant. Moreover, we have the following lemma.

Lemma 2.2.4. *If $f \in \mathbb{C}[\vec{x}]$ is χ_k-invariant, then f_{χ_k} divides f.*

Proof. It suffices to prove that $(L_H)^k$ divides f for each $H \in \mathcal{H}(G)$ since $\mathbb{C}[\vec{x}]$ is a unique factorization domain and each L_H is irreducible. Let $s \in G$ be a pseudo-reflection with fixed hyperplane $H \in \mathcal{H}(G)$. Since $f \in \mathbb{C}[\vec{x}]^G_\chi$,

$$sf = \chi_k(s)f = \det(s)^{-k}f = \zeta_s^{-k}f$$

where ζ_s is the exceptional eigenvalue of s.

By changing coordinates if necessary, we may assume that H is defined by $x_n = 0$ and the matrix for s diagonal with ones in all but the last entry which is ζ_s. In these coordinates, we can write

$$sf(\vec{x}) = f(s^{-1}\vec{x}) = f(x_1, \ldots, x_{n-1}, \zeta_s^{-1}x_n).$$

By our previous argument we now have

$$sf(\vec{x}) = f(x_1, \ldots, x_{n-1}, \zeta_s^{-1}x_n) = \zeta_s^{-k}f(x_1, \ldots, x_n).$$

We must now conclude that x_n^k divides $f(\vec{x})$. Thus $(L_H)^k$ divides f. $\square$

We are now ready to prove the main theorem of this section.

Theorem 2.2.5. *Let $G \subset \mathrm{GL}(\mathbb{C})$ be a finite pseudo-reflection group with G^1 and f_{χ_k} defined as above. Then*

$$\mathbb{C}[\vec{x}]^{G^1} = \bigoplus_{k=0}^{m-1} f_{\chi_k}\mathbb{C}[\vec{x}]^G,$$

where m is the order of the cyclic group G/G^1 and $\mathbb{C}[\vec{x}]^G$ is the invariant ring of G.

Proof. [10, pages 140–141] This proof will proceed by showing mutual containment.

One containment is fairly obvious. Let $f \in \bigoplus_{k=0}^{m-1} f_{\chi_k}\mathbb{C}[\vec{x}]^G$. Then $f = \sum_{k=0}^{m-1} f_{\chi_k}f_k$ where each $f_k \in \mathbb{C}[\vec{x}]^G$. Given any $s \in G^1$,

$$sf = \sum_{k=0}^{m-1}(sf_{\chi_k})(sf_k).$$

For each term in this sum,

$$(sf_{\chi_k})(sf_k) = (\chi_k(s)f_{\chi_k})f_k = (\det(s)^{-k}f_{\chi_k})f_k = (f_{\chi_k})f_k$$

since $\det(s) = 1$ and $f_k \in \mathbb{C}[\vec{x}]^G$. Thus $sf = f$ for all $s \in G^1$, so $f \in \mathbb{C}[\vec{x}]^{G^1}$. This means that we have

$$\bigoplus_{k=0}^{m-1} f_{\chi_k}\mathbb{C}[\vec{x}]^G \subset \mathbb{C}[\vec{x}]^{G^1}.$$

Now let $f \in \mathbb{C}[\vec{x}]^{G^1}$. Fix $t \in G$ where $G/G^1 = \langle t\rangle$ and define

$$g_k(\vec{x}) = \sum_{i=0}^{m-1} \det(t)^{-ik} f(t^i\vec{x}).$$

Then we can rewrite

$$f(\vec{x}) = \frac{1}{m}\sum_{k=0}^{m-1} g_k(\vec{x}).$$

The goal here is to show that g_k is χ_k-invariant for each $0 \leq k \leq m-1$ since this would tell us that f_{χ_k} divides g_k for each k.

We need only examine the behavior of the g_k's under the various powers of t since every element in G can be written as a power of t times an element with determinant 1. In other words, for every $s \in G$, we have $s = t^j g$ where $g \in G^1$ and $0 \leq j \leq m-1$, so for each $s \in G$, $\chi_k(s) = \chi_k(t^j)$ and $sf = t^j f$ for some j.

Consider the action on g_k by t^j for some $0 \leq j \leq m-1$. Since $\det(t)^{-ik} = \zeta_t^{-ik}$, we have

$$g_k(\vec{x}) = \sum_{i=0}^{m-1} \zeta_t^{-ik} f(t^i\vec{x}),$$

which means

$$t^j g_k = \sum_{i=0}^{m-1} \zeta_t^{-ik} f(t^{i-j}\vec{x}).$$

We reindex letting $\ell = i - j$ to get

$$t^j g_k = \sum_{\ell=0}^{m-1} \zeta_t^{-k(\ell+j)} f(t^\ell\vec{x}) = \zeta_t^{-kj}\sum_{\ell=0}^{m-1} \zeta_t^{-\ell k} f(t^\ell\vec{x}) = \chi_k(t^j)g_k.$$

This means that g_k is χ_k-invariant for every k, so by Lemma 1 we know that $g_k = f_{\chi_k}\hat{g}_k$ for some $\hat{g}_k \in \mathbb{C}[\vec{x}]$. Thus for any $s \in G$,

$$\chi_k(s)f_{\chi_k}\hat{g}_k = \chi_k(s)g_k = sg_k = (sf_{\chi_k})(s\hat{g}_k) = \chi_k(s)f_{\chi_k}(s\hat{g}_k),$$

which means that $\hat{g}_k = s\hat{g}_k$ or $\hat{g}_k \in \mathbb{C}[\vec{x}]^G$. Therefore, $f \in \bigoplus_{k=0}^{m-1} f_{\chi_k}\mathbb{C}[\vec{x}]^G$, so

$$\mathbb{C}[\vec{x}]^{G^1} \subset \bigoplus_{k=0}^{m-1} f_{\chi_k}\mathbb{C}[\vec{x}]^G.$$

$\square$

Chapter 3

Orbit Ideals and Hilbert Functions

3.1 The Orbit Ideal of G

In this section, $G \subset \mathrm{GL}(\mathbb{C}^n)$ will be a finite pseudo-reflection group. Let $R = \mathbb{C}[\vec{x}]$, and let $\theta_1, \ldots, \theta_n$ be the primary invariants of G so that $\mathbb{C}[\vec{x}]^G = \mathbb{C}[\theta_1, \ldots, \theta_n] \subset R$. Define $d_i = \deg(\theta_i)$. We will consider $\mathbb{C}^n$ embedded in $\mathbb{CP}^n$ via

$$(p_1, \ldots, p_n) \mapsto (1, p_1, \ldots, p_n).$$

Definition 3.1.1. Let $p \in \mathbb{C}^n$ be a single point. The G-orbit of the point p is the finite set $O_G(p) = \{g(p) \mid g \in G\} \subset \mathbb{CP}^n$.

We will assume we have chosen $p \in \mathbb{C}^n \setminus \mathcal{H}(G)$, so that $|O_G(p)| = |G|$.

Let I_p be the ideal generated by the homogeneous polynomials in $R[x_0]$ which vanish on $O_G(p)$.

Theorem 3.1.2. *$I_p = (\gamma_1, \ldots, \gamma_n)$, where $\gamma_i = \theta_i - \theta_i(p)x_0^{d_i}$.*

Proof. We would like to show that the sequence of γ_i's is regular, meaning each γ_i is not a zero divisor in the ring $R[x_0]/(\gamma_1, \ldots, \gamma_{i-1})$. A theorem of Macaulay states that if a ring A is Cohen-Macaulay, showing a sequence $a_1, \ldots, a_r \in A$ is regular is equivalent to showing $\dim(A/(a_1, \ldots, a_r)) = \dim(A) - r$. [9, page 165] We will use this theorem to prove that the sequence $\gamma_1, \ldots, \gamma_n, x_0$ is regular by showing that

$$\dim(R[x_0]/(I_p, x_0)) = 0 = \dim(R[x_0]) - (n+1).$$

We first observe that $\theta_1, \ldots, \theta_n$ is a regular sequence, because any system of parameters for a Cohen-Macaulay ring forms a regular sequence. [9, page 166] Since the θ_i's are a system of parameters for $\mathbb{C}[\vec{x}]^G$ which is Cohen-Macaulay, we know that they are regular.

Now look at $R[x_0]/(I_p, x_0)$. When we set $x_0 = 0$ in each of the generators $\gamma_i = \theta_i - \theta_i(p)x_0^{d_i}$ we get simply θ_i, so $R[x_0]/(I_p, x_0) \cong R/(\theta_1, \dots, \theta_n)$. Since the θ_i's are regular, $\dim(R/(\theta_1, \dots, \theta_n)) = 0 = \dim(R[x_0]/(I_p, x_0))$. Therefore the sequence $\gamma_1, \dots, \gamma_n, x_0$ is regular, so the subsequence consisting of only the γ_i's is also regular.

This means that $\dim(R[x_0]/I_p) = \dim(R[x_0]) - n = 1$. By Macaulay's Unmixedness theorem, the fact that $R[x_0]$ is Cohen-Macaulay and I_p is generated by a regular sequence implies that all the associated primes of I_p are minimal and their zero sets are projective points. Therefore I_p has a primary decomposition of the form

$$I_p = \bigcap_{q \in Z(I_p)} m(q)^{e_q}.$$

Since I_p clearly has all the points in $O_G(p)$ as elements of its zero set, we can write

$$I_p = \left(\bigcap_{g \in G} m(g(p))^{e_g} \right) \cap \left(\bigcap_q m(q)^{e_q} \right)$$

where the ideal $m(g(p))$ is the ideal for the point $g(p)$ and the ideals $m(q)$ are the ideals for all additional points in the zero set of I_p. By a theorem of Serre, $\sum e_g + \sum e_q = \deg(R[x_0]/I_p)$ [8, page 100]. Here $\deg(R[x_0]/I_p)$ is defined as $(\dim(R[x_0]/I_p) - 1)!$ times the coefficient of the leading term of the Hilbert polynomial. (This will be the term with exponent equal to $\dim(R[x_0]/I_p) - 1$.) In our case we know that $\dim(R[x_0]/I_p) - 1 = 0$, so we are looking for the constant term, which is the leading term, of the Hilbert polynomial.

Since the γ_i's form a regular sequence, a standard argument, given below, shows that this constant term is $\prod_i \deg(\gamma_i) = \prod_i d_i$. From Serre's theorem and our earlier result about finite pseudo-reflection groups, we now have

$$I_p = \left(\bigcap_{g \in G} m(g(p))^{e_g} \right) \cap \left(\bigcap_q m(q)^{e_q} \right)$$

where

$$\sum e_g + \sum e_q = \deg(R[x_0]/I_p) = \prod_i d_i = |G|,$$

and we can conclude that for each $g \in G$, $e_g = 1$ while $e_q = 0$ for all q. Thus the zero set of I_p contains exactly the points in $O_G(p)$, which makes I_p the ideal describing $O_G(p)$ as claimed. $\square$

The standard argument used to find the constant term of the Hilbert polynomial goes as follows:

To find the Hilbert polynomial, we first determine the Hilbert series of $R[x_0]/I_p$. Since we know the Hilbert polynomial is a constant, there will be some exponent past which all coefficients of the Hilbert series will equal $\deg(R[x_0]/I_p)$.

Let S be a graded polynomial ring and consider the sequence given by

$$0 \longrightarrow S(-d) \xrightarrow{f} S \longrightarrow S/(f) \longrightarrow 0$$

where $\deg(f) = d$ and $S(-d)$ denotes the ring S with a shift by d in the grading: $S(-d)_e = S_{e-d}$. If f is not a zero-divisor in S, then this sequence is exact. Thus

$$H_{S/(f)}(\ell) = H_S(\ell) - H_S(\ell - d).$$

Multiplying by z^ℓ and summing over ℓ we get

$$\begin{aligned}\Phi_{S/(f)}(z) &= \sum_{\ell \geq 0} H_{S/(f)}(\ell) z^\ell = \sum_{\ell \geq 0} H_S(\ell) z^\ell - z^d \sum_{\ell \geq 0} H_S(\ell - d) z^{\ell - d} \\ &= (1 - z^d) \sum_{\ell \geq 0} H_S(\ell) z^\ell = (1 - z^d)\Phi_S(z).\end{aligned}$$

However, when $S = R[x_0]$, $H_S(\ell) = \binom{n+\ell}{\ell}$. Therefore

$$\Phi_{R[x_0]}(z) = \sum_{\ell \geq 0} \binom{n+\ell}{\ell} z^\ell = \frac{1}{(1-z)^{n+1}}.$$

Thus by induction we have

$$\begin{aligned}\Phi_{R[x_0]/I_p}(z) &= \frac{\prod_{i=1}^n (1 - z^{d_i})}{(1-z)^{n+1}} = \frac{1}{1-z} \prod_{i=1}^n (1 + z + \cdots + z^{d_i - 1}) \\ &= \left[\prod_{i=1}^n (1 + z + \cdots + z^{d_i - 1})\right]\left[1 + z + z^2 + \ldots\right] = \sum_{\ell \geq 0} a_\ell z^\ell.\end{aligned}$$

We would like to show that there is some ℓ_0 such that for all $\ell \geq \ell_0$, $a_\ell = \prod_i d_i$. Our proof will proceed by induction on n. Let $n = 1$. Then

$$\begin{aligned}\Phi_{R[x_0]/I_p}(z) &= (1 + z + \cdots + z^{d_1 - 1})(1 + z + z^2 + \ldots) \\ &= 1 + 2z + 3z^2 + \cdots + d_1 z^{d_1 - 1} + d_1 z^{d_1} + \ldots\end{aligned}$$

so past $\ell_0 = d_1 - 1$, we have $a_\ell = d_1$.

Now suppose the result for $n-1$. Then for n we have

$$\Phi_{R[x_0]/I_p}(z) = \frac{\prod_{i=1}^{n}(1-z^{d_i})}{(1-z)^{n+1}} = \frac{\prod_{i=1}^{n-1}(1-z^{d_i})}{(1-z)^n}\cdot\frac{(1-z^{d_n})}{(1-z)}$$
$$= (1+\cdots+(\textstyle\prod_{i=1}^{n-1} d_i)z^{\ell} + (\prod_{i=1}^{n-1} d_i)z^{\ell+1}+\dots)(1+z+\cdots+z^{d_n-1}).$$

If we let ℓ be large enough, then the z^ℓ term will be

$$\sum_{k=0}^{d_n-1}\left(\prod_{i=1}^{n-1} d_i\right)z^{\ell-k}z^k = \left(\prod_{i=1}^{n-1} d_i\right)\sum_{k=0}^{d_n-1} z^\ell = \left(\prod_{i=1}^{n} d_i\right)z^\ell.$$

Thus for large enough ℓ the coefficient on z^ℓ is always $\prod_{i=1}^n d_i$, so the Hilbert polynomial of $R[x_0]/I_p$ is just $\prod_i d_i$. Therefore $\deg(R[x_0]/I_p) = \prod_i d_i$.

3.2 Hilbert Functions of G and G^1

The proof of the last section's final theorem computed the Hilbert function of a finite pseudo-reflection group in terms of the degrees of its primary invariants. In this section we will compute for small values of n the Hilbert functions of the three main families of reflection groups as well as the Hilbert functions of their subgroups of determinant 1. Since we are dealing only with reflection groups, the subgroup of determinant 1, G^1, is the subgroup of all the even length elements and has order equal to half the order of G. These Hilbert functions were computed using the computer algebra system CoCoA. [2]

When $G = A_n$, we know that $\mathbb{C}[\vec{x}]^G = \mathbb{C}[\sigma_1,\dots,\sigma_n]$ where σ_i is the i-th symmetric polynomial; so for all i, we have $d_i = i$. It follows that

$$\Phi_{A_n}(z) = \left[\prod_i(1+z+\cdots+z^{i-1})\right](1+z+z^2+\dots).$$

When $G = A_n$, G^1 is the group whose elements are made up of even numbers of transpositions.

For $n=3$ we have:

$$\begin{aligned}\Phi_{A_3}(z) &= 1+3z+5z^2+6z^3+6z^4+\dots\\ \Phi_{G^1}(z) &= 1+3z+3z^2+\dots\end{aligned}$$

For $n=4$ we have:

$$\begin{aligned}\Phi_{A_4}(z) &= 1+4z+9z^2+15z^3+20z^4+23z^5+24z^6+24z^7+\dots\\ \Phi_{G^1}(z) &= 1+4z+9z^2+12z^3+12z^4+\dots\end{aligned}$$

For $n = 5$ we have:

$$\begin{aligned}\Phi_{A_5}(z) = \;& 1 + 5z + 14z^2 + 29z^3 + 49z^4 + 71z^5 + 91z^6 + 106z^7 + 115z^8 \\ & +119z^9 + 120z^{10} + \dots \\ \Phi_{G^1}(z) = \;& 1 + 5z + 14z^2 + 29z^3 + 49z^4 + 60z^5 + 60z^6 + \dots\end{aligned}$$

When $G = B_n$, we know $\mathbb{C}[\vec{x}]^G = \mathbb{C}[\tau_1, \dots, \tau_n]$ where $\tau_i = \sigma_i(x_1^2, \dots, x_n^2)$; so for all i, we have $d_i = 2i$. This means that

$$\Phi_{B_n}(z) = \left[\prod_i (1 + z + \cdots + z^{2i-1})\right](1 + z + z^2 + \dots).$$

When $G = B_n$, G^1 is the group whose elements are made up of w transpositions and x sign changes where $w + x$ is even.

For $n = 3$ we have:

$$\begin{aligned}\Phi_{B_3}(z) = \;& 1 + 4z + 9z^2 + 16z^3 + 24z^4 + 32z^5 + 39z^6 + 44z^7 + 47z^8 \\ & +48z^9 + \dots \\ \Phi_{G^1}(z) = \;& 1 + 4z + 9z^2 + 16z^3 + 24z^4 + \dots\end{aligned}$$

For $n = 4$ we have:

$$\begin{aligned}\Phi_{B_4}(z) = \;& 1 + 5z + 14z^2 + 30z^3 + 54z^4 + 86z^5 + 125z^6 + 169z^7 + 215z^8 + 259z^9 \\ & +298z^{10} + 330z^{11} + 354z^{12} + 370z^{13} + 379z^{14} + 383z^{15} + 384z^{16} + \dots \\ \Phi_{G^1}(z) = \;& 1 + 5z + 14z^2 + 30z^3 + 54z^4 + 86z^5 + 125z^6 + 169z^7 + 192z^8 + \dots\end{aligned}$$

When $G = D_n$, we know $\mathbb{C}[\vec{x}]^G = \mathbb{C}[\tau_1, \dots, \tau_{n-1}, \sigma_n]$; so for $i = 1, \dots, n-1$, we have $d_i = 2i$. When $i = n$, we get $d_i = n$. This means that

$$\Phi_{D_n}(z) = \left[\prod_{i=1}^{n-1} (1 + z + \cdots + z^{2i-1})\right](1 + z + \cdots + z^{n-1})(1 + z + z^2 + \dots).$$

When $G = D_n$, G^1 is the group whose elements are made up of u transpositions and $2v$ sign changes where $u + v$ is even.

For $n = 4$ we have:

$$\begin{aligned}\Phi_{D_4}(z) = \;& 1 + 5z + 14z^2 + 30z^3 + 53z^4 + 81z^5 + 111z^6 + 139z^7 + 162z^8 + 178z^9 \\ & +187z^{10} + 191z^{11} + 192z^{12} + \dots \\ \Phi_{G^1}(z) = \;& 1 + 5z + 14z^2 + 30z^3 + 53z^4 + 81z^5 + 96z^6 + \dots\end{aligned}$$

For each group G, comparing $\Phi_G(z)$ with $\Phi_{G^1}(z)$ we see that the Hilbert function of G^1 is term for term exactly the same as the Hilbert function of G until its final term. There the coefficient of $\Phi_{G^1}(z)$ must equal the order of G^1 which is half the order of G. Put simply, the Hilbert function of G^1 "keeps up" with the Hilbert function of G as long as it can. It is our conjecture that this is the case in general for all reflection groups. For instance, in the case of A_n, this means that if a polynomial of degree less than $\binom{n}{2}/2$ vanishes on a generic orbit of some even permutation under the alternating group, then it vanishes on a generic orbit under the full symmetric group.

Chapter 4

The Covariant Ring

4.1 Covariant Rings and Harmonics

In this section, G will be a finite pseudo-reflection group and R will denote the polynomial ring $\mathbb{C}[\vec{x}]$. We will denote the degree d piece of R by R_d.

Definition 4.1.1. Let I be the graded ideal generated by all homogeneous invariants of G which have positive degree. Then $R_G = R/I$ is the *covariant ring* of G.

Since $\mathbb{C}[\vec{x}]^G = \mathbb{C}[\theta_1, \ldots, \theta_n]$, we know that $I = (\theta_1, \ldots, \theta_n)$. From the induction argument on exact sequences in the last chapter, we can use this information to easily compute the Hilbert series of the covariant ring of G. Recall from the previous chapter that $\gamma_i = \theta_i - \theta_i(p)x_o^{d_i}$. The sequence

$$0 \longrightarrow R[x_o]/(\gamma_i)(-1) \xrightarrow{x_0} R[x_o]/(\gamma_i) \longrightarrow R[x_0]/(\gamma_i, x_0) \longrightarrow 0$$

is exact and $R[x_0]/(\gamma_i, x_0) = R/I = R_G$, so

$$\Phi_{R_G}(z) = (1-z)\Phi_{R[x_0]/(\gamma_i)} = (1-z)\frac{\prod_i (1-z^{d_i})}{(1-z)^{n+1}} = \prod_i \frac{1-z^{d_i}}{1-z}.$$

While knowing the Hilbert function of the covariant ring is helpful, to really get our hands around R_G, it is useful to have another way to compute it. To do so, we will need the following definitions.

Definition 4.1.2. Let $f, g \in R$. We define $f(\partial) = f(\partial_1, \ldots, \partial_n)$ where $\partial_i = \frac{\partial}{\partial x_i}$. Then the action of $f(\partial)$ on $g(x)$ is a linear extension of the action of a monomial $\partial^q = \partial_1^{q_1} \cdots \partial_n^{q_n}$ on another monomial $x^p = x_1^{p_1} \cdots x_n^{p_n}$. This action is defined to be

$$\partial^q(x^p) = \partial_1^{q_1} \cdots \partial_n^{q_n}(x_1^{p_1} \cdots x_n^{p_n}) = \prod_i p_i(p_i - 1) \cdots (p_i - q_i + 1)x_i^{p_i - q_i}.$$

Definition 4.1.3. For any homogeneous ideal $J = (f_1, \ldots, f_n) \subset R$, we call the *killed space* of J

$$\mathcal{K}(J) = \{g \in R \mid f(\partial)(g) = 0 \text{ for all } f \in J\}.$$

Note that $\mathcal{K}(J) \subset R$ inherits the structure of a graded ring, but probably is not an ideal of R. It is easy to prove that this definition of $\mathcal{K}(J)$ is equivalent to $\mathcal{K}(J) = \{g \in R \mid f_i(\partial)(g) = 0 \text{ for all } i\}$.

Our definition of one polynomial acting on another as a differential operator allows us to define a pairing on each graded piece of R.

Definition 4.1.4. Let $\langle\ ,\ \rangle : R_d \times R_d \to \mathbb{C}$ where for any $f, g \in R_d$, $\langle f, g\rangle = f(\partial)(g)$.

The next two lemmas follow [5].

Lemma 4.1.5. *The pairing defined above is symmetric and nonsingular, i.e.* $\langle f, g\rangle = \langle g, f\rangle$, $\langle f, g\rangle = 0$ *for all* $f \in R_d$ *iff* $g = 0$, *and* $\langle f, g\rangle = 0$ *for all* $g \in R_d$ *iff* $f = 0$.

Proof. Let $f(x) = \sum_I \alpha_I x^I$ and $g = \sum_J \beta_J x^J$, where $I = (i_1, \ldots, i_n)$ and $J = (j_1, \ldots, j_n)$ such that $\sum_k i_k = \sum_k j_k = d$. Then $\langle f, g\rangle = \left\langle \sum_I \alpha_I x^I, \sum_J \beta_J x^J \right\rangle = \sum_{I,J} \alpha_I \beta_J \left\langle x^I, x^J \right\rangle$. Clearly the only nonzero terms are those where $I = J$, so the sum becomes

$$\sum_I \alpha_I \beta_I \left\langle x^I, x^I \right\rangle = \sum_I (I!) \alpha_I \beta_I.$$

From this calculation, it is easy to see that the pairing is symmetric.

Now suppose that $f \neq 0$. Then by our previous calculation, $\langle f, f\rangle \neq 0$. Thus our pairing $\langle\ ,\ \rangle$ is nonsingular. $\square$

With this inner product on each graded piece of R we can take orthogonal complements of sets within each R_d.

Definition 4.1.6. Let S be a set in R_d. Then

$$S^\perp = \{g \in R_d | \langle f, g\rangle = 0 \text{ for all } f \in S\}.$$

Lemma 4.1.7. *Let* $J \subset R$ *be a homogeneous ideal. Then*

$$(\mathcal{K}(J))_d = (J_d)^\perp.$$

Proof. It is easy to see that we have the inclusion $(\mathcal{K}(J))_d \subset (J_d)^\perp$, because if $g \in (\mathcal{K}(J))_d$ then $\langle f, g\rangle = 0$ for all $f \in J_d$ since $J_d \subset J$.

Now suppose $g \in (J_d)^\perp$. Choose any $f \in J$. Without loss of generality we may choose f to be one of the homogeneous generators of J. We need to show $\langle f, g\rangle = 0$. Since $g \in (J_d)^\perp$, $\deg(g) = d$. Thus if $\deg(f) \geq d$ then we are done, so let $\deg(f) < d$.

Set $\alpha = (a_1, \ldots, a_n)$ where $\sum_{i=1}^n a_i = d - \deg(f)$. This makes $\deg(x^\alpha f) = d$ with $x^\alpha f \in J_d$. Thus for all such monomials x^α,

$$\begin{aligned} 0 = \langle x^\alpha f, g\rangle &= (x^\alpha f)(\partial)(g) = x^\alpha(\partial)(f(\partial)(g)) \\ &= \langle x^\alpha, f(\partial)(g)\rangle = \langle x^\alpha, \langle f, g\rangle\rangle . \end{aligned}$$

Since $x^\alpha, \langle f, g\rangle \in R_{d-\deg(f)}$ and the pairing is nonsingular, we must have $\langle f, g\rangle = 0$. This means $g \in (\mathcal{K}(J))_d$ so $(J_d)^\perp \subset (\mathcal{K}(J))_d$, completing the proof. □

In the special case where $J = (\theta_1, \ldots, \theta_n) = I$, we call $\mathcal{K}(I) = \mathcal{K}(\theta_1, \ldots, \theta_n)$ the *harmonics* of G denoted H_G. The name comes from the fact that any orthogonal matrix preserves length, leaving the polynomial $x_1^2 + \cdots + x_n^2$ invariant. This means that in the Euclidean case, if a polynomial g is in H_G, we know g satisfies Laplace's equation: $\sum_{i=1}^n \frac{\partial^2 g}{\partial^2 x_i^2} = 0$.

Theorem 4.1.8. *$H_G = R_G$, i.e. the ring of harmonics is isomorphic to the ring of covariants.*

Proof. By definition, $H_G = \mathcal{K}(I)$, so by the above lemma, $(H_G)_d = (\mathcal{K}(I))_d = (I_d)^\perp$. Since the pairing $\langle\ ,\ \rangle : R_d \times R_d \to \mathbb{C}$ is nonsingular, R_d decomposes as

$$R_d = I_d \oplus (I_d)^\perp = I_d \oplus (H_G)_d.$$

This means that

$$(H_G)_d = R_d/I_d = (R/I)_d = (R_G)_d,$$

so taking the direct sum over degrees we see that $H_G = R/I = R_G$. □

This theorem allows us to calculate the covariant ring of a group G by computing the set of polynomials killed by the generators of I.

4.2 The Harmonics and Ω

Although we now have a fairly straightforward way of computing H_G, there is an even easier way to obtain H_G explicitly. Recall that in section 2.2 where we discussed the invariant ring of G^1, we defined the concept of a χ_k-invariant function and constructed a polynomial f_{χ_k} which divided all other χ_k-invariant functions. In this section we will restrict ourselves to the case where $k = 1$, the so-called "skew" invariants.

Definition 4.2.1. To simplify our notation we define $\Omega = f_{\chi_1}$.

Remember that $\deg(\Omega)$ is the number of reflections in G.

Lemma 4.2.2. *$\Omega \in H_G$.*

Proof. Let $f \in I$. Then for any $g \in G$,

$$g(f(\partial)(\Omega)) = f(g^{-1}\partial)(g\Omega) = f(\partial)(\det(g)^{-1}\Omega) = \det(g)^{-1}f(\partial)(\Omega)$$

since f is invariant and Ω is skew. This means that $f(\partial)(\Omega)$ is skew. Thus we have $\Omega | f(\partial)(\Omega)$. However since $\deg(f(\partial)(\Omega)) < \deg(\Omega)$, we must have $f(\partial)(\Omega) = 0$. □

Lemma 4.2.3. *Let $f \in R$. If $f(\partial)(\Omega) = 0$ then $f \in I$.*

Proof. From [7].

Let $f \in R$ be chosen so that $f(\partial)(\Omega) = 0$. Since the ring R/I is finite dimensional, we know $f \in I$ if its degree is high enough. The proof will proceed by downward induction on degree, so assume that the result holds for degree greater than $\deg(f)$.

Let $s \in G$ be a pseudo-reflection and L_{H_s} be the linear polynomial defining H_s. As in Lemma 2.2.4, we know $sL_{H_s} = \det(s)L_{H_s}$. Since the polynomial $L_{H_s}f$ has degree greater than the degree of f and $L_{H_s}f(\partial)(\Omega) = L_{H_s}(\partial)f(\partial)(\Omega) = 0$ we have $L_{H_s}f \in I$.

This means we can write $L_{H_s}f = f_1\theta_i + \cdots + f_n\theta_n$ where each $f_i \in R$. Applying s to this we get

$$(\det(s)L_{H_s})(sf) = (sL_{H_s})(sf) = s(L_{H_s}f) = (sf_1)\theta_1 + \cdots + (sf_n)\theta_n.$$

From these calculations, we see that

$$\det(s)sf - f = \frac{sf_1 - f_1}{L_{H_s}}\theta_1 + \cdots + \frac{sf_n - f_n}{L_{H_s}}\theta_n$$

with $\frac{sf_i - f_i}{L_{H_s}} \in R$ for each i, since $(sf_i - f_i)(H_s) = 0$. This means that we have $\det(s)sf - f \in I$ so

$$f \equiv \det(s)sf \mod I$$

for all pseudo-reflections s. Since G is generated by pseudo-reflections, we have

$$f \equiv \det(g)gf \mod I$$

for all $g \in G$.

Since this last equivalence holds for all $g \in G$, we can sum both sides over G and divide by $|G|$. This gives us

$$f \equiv \frac{1}{|G|}\sum_{g\in G}\det(g)gf \mod I.$$

Given this relationship, watch what happens to f when we act on it with an element $h \in G$.

$$\begin{aligned} hf &\equiv \frac{1}{|G|}\sum_{g\in G}\det(g)ghf \equiv \det(h)^{-1}\frac{1}{|G|}\sum_{g\in G}\det(hg)ghf \\ &\equiv \det(h)^{-1}\frac{1}{|G|}\sum_{g\in G}\det(g)gf \equiv \det(h)^{-1}f \mod I. \end{aligned}$$

This shows that f is a skew invariant modulo I, so we know that

$$f \equiv b\,\Omega \mod I$$

for some $b \in R$. If $\deg(b) > 0$ then $\deg(f) > \deg(\Omega) = \sum_i (d_i - 1)$ which is the highest degree of anything in R/I so $f = b\,\Omega \in I$. Thus if $\deg(b) > 0$ we are done, so assume $b \in \mathbb{C}$.

Then since $f = b\,\Omega + i$ for some $i \in I$ we have

$$f(\partial)(\Omega) = b\,\Omega(\partial)(\Omega) + i(\partial)(\Omega) = b\,\Omega(\partial)(\Omega) = 0,$$

but since $\Omega(\partial)(\Omega) \neq 0$, we must have $b = 0$. Therefore $f \equiv 0 \mod I$ as required. □

Definition 4.2.4. Let $\partial^\bullet\Omega$ be the linear span of partial derivatives of Ω.

Theorem 4.2.5. $H_G = \partial^\bullet \Omega$.

Proof. To prove this result, we will show mutual containment. One way is easy. Since $\Omega \in H_G$ and for each $f \in I$, $f(\partial)(\Omega) = 0$, we know that each element of I must also kill each of Ω's derivatives. This means that we have $\partial^\bullet \Omega \subset H_G$.

To go the other way, let $f \in (\partial^\bullet \Omega_d)^\perp$. Recalling that $\deg(\Omega) = r =$ the number of reflections in G, we know that for all $x^\alpha \in R_{r-d}$

$$0 = f(\partial)x^\alpha(\partial)(\Omega) = (fx^\alpha)(\partial)(\Omega) = x^\alpha(\partial)f(\partial)(\Omega).$$

Since $f(\partial)(\Omega) \in R_{r-d}$, this is equivalent to saying

$$\langle x^\alpha, f(\partial)(\Omega)\rangle = 0$$

for all $x^\alpha \in R_{r-d}$. Since our pairing is nonsingular, this means we must have $f(\partial)(\Omega) = 0$ so by our last lemma, $f \in I_d$. Thus $(\partial^\bullet \Omega_d)^\perp \subset I_d$ or

$$(H_G)_d = (I_d)^\perp \subset \partial^\bullet \Omega_d.$$

Summing over d we get $H_G = \partial^\bullet \Omega$ as claimed. □

4.3 The Covariant Ring as the Regular Representation

It is possible to look at the covariant ring of G as a vector space over $\mathbb{C}$. Since G is a pseudo-reflection group, there is a very special way in which G acts on R/I.

Definition 4.3.1. The *regular representation* of a group G is a vector space with a basis $\{v_g \mid g \in G\}$ where G acts on V by $hv_g = v_{hg}$ for all $h, g \in G$.

Recall the definition of $O_G(p)$ as the orbit of a point p under G embedded in projective space. In the last chapter we proved that the ideal of polynomials vanishing on $O_G(p)$ is precisely $I_p = (\theta_i(x) - \theta_i(p)x_0^{d_i}) \subset R[x_0]$. The ring $A = R[x_0]/I_p$ is graded, so we can write it as $A = A_0 \oplus A_1 \oplus A_2 \oplus \dots$ where A_ℓ is the piece of degree ℓ.

We can find some $\phi_e(x) \in A_d$, of minimal degree d, such that

$$\phi_e(q) = \begin{cases} 1 & q = p \\ 0 & q \in O_G(p) \setminus \{p\}. \end{cases}$$

For each $g \in G$, define $\phi_g(x) = \phi_e(g^{-1}x)$. Note that all the ϕ_g's live in A_d. Also note that for each $g \in G$, because the ϕ_g's are of minimal degree we know that $x_0 \nmid \phi_g(x)$.

Lemma 4.3.2. *The set $\{\phi_g(x) \mid g \in G\}$ forms a basis for A_d.*

Proof. Let $f \in R$. Then

$$f(q) - \sum_{g\in G} f(q)\phi_g(q) = 0 \text{ for all } q \in O_G(p).$$

This means that

$$f(x) = \sum_{g\in G} f(x)\phi_g(x) \mod I_p.$$

□

We can see that $\dim(A_d) = |G|$ and that A_d is the regular representation of G.

Theorem 4.3.3. *The ring of covariants, R_G, is the regular representation of G.*

Proof. Let the map $\psi : A_d \to R_G$ be defined by $\phi_g(x) \mapsto \phi_g(x)|_{x_0=0} \mod I$. Since $\psi(\gamma_i) = \theta_i$ for each i and x_0 does not divide any of the basis vectors of A_d, we know that the map ψ is well-defined and injective.

From the previous lemma we know that $\dim(A_d) = |G|$. However, evaluating the Hilbert series of R_G at $z = 1$, we get that

$$\dim(R_G) = \Phi_{R_G}(1) = \prod_i (1 + 1^1 + 1^2 + \cdots + 1^{d_i-1}) = \prod_i d_i = |G|.$$

Thus ψ is an injective map between two vector spaces of equal dimension which means ψ is an isomorphism. This gives us $R_G \cong A_d$ so since A_d is the regular representation of G, so is R_G. □

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