Multi-cyclic Polytopes

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Abstract

For an arbitrary real-valued representation ρ of a cyclic group $\mathbb{Z}/n\mathbb{Z}$, we define a polytope $P = \mathrm{conv}\{\rho(a) \mid a \in \mathbb{Z}/n\mathbb{Z}\}$, which we refer to as a *multi-cyclic polytope*. In studying the face structure of this family of polytopes we first present a connection to generalized Vandermonde matrices and Schur functions. A duality is later proved between various representations for any fixed cyclic group and is applied to classify certain sets of these polytopes. We further reduce the problem of their face structure to a problem concerning minimal relations among roots of unity over $\mathbb{R}^+$.

Introduction

0.1 Representations of Cyclic Groups

For any finite group G, a *representation* ρ is a homomorphism $\rho : G \to GL(V)$, into the set of all invertible linear maps on a vector space V over $\mathbb{C}$. For $G = \mathbb{Z}/n\mathbb{Z}$, the *characters*, homomorphisms $\chi_k \colon G \to \mathbb{C}^*$, are representations. These must have the form $\chi_k(a) = e^{\frac{2\pi\imath}{n}ka}$ where $k \in \{0, 1, \ldots, n-1\}$. It turns out that all representations of $\mathbb{Z}/n\mathbb{Z}$ are direct sums of characters.

The fundamental real-valued representations ρ of $\mathbb{Z}/n\mathbb{Z}$ are:

$$\text{for } k = 0: \;\; \rho(a) = \chi_0(a) = e^0 = 1 \tag{1}$$

$$\text{for } k = \frac{n}{2}: \;\; \rho(a) = \chi_{\frac{n}{2}}(a) = e^{\pi i a} = (-1)^a \tag{2}$$

$$\text{for other } k : \rho(a) = \begin{pmatrix} \cos(\frac{2\pi ka}{n}) & -\sin(\frac{2\pi ka}{n}) \\ \sin(\frac{2\pi ka}{n}) & \cos(\frac{2\pi ka}{n}) \end{pmatrix}. \tag{3}$$

Thinking of (3) as a representation in $\mathbb{C}^2$, we could, by a linear change of coordinates, get

$$\begin{pmatrix} \chi_k(a) & 0 \\ 0 & \chi_{-k}(a) \end{pmatrix}.$$

Note that $\chi_{-k} = \overline{\chi_k}$, so that both χ_k and its conjugate appear as part of the representation. So in (3), $\rho(a) = \chi_k(a) \oplus \chi_{-k}(a)$.

For an arbitrary real representation then, we have

$$\rho(a) = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_m \end{pmatrix} \in \mathbb{R}^{d \times d} \tag{4}$$

where B_j is of the form of (1), (2), or (3).

0.2 Polytopes

We now present an introduction of the necessary aspects of the theory of polytopes for this paper. For a more thorough treatment of the material, see [8].
Definition. The *convex hull* of $K \subseteq \mathbb{R}^d$ is the smallest convex set containing K:

$$\operatorname{conv}(K) := \bigcap \{K' \subset \mathbb{R}^d \mid K \subset K', K' \text{ convex}\}.$$

A *polytope* P in $\mathbb{R}^d$ is the convex hull of a finite set of points $V \in \mathbb{R}^d$. If the points in V span an affine space of dimension k, then we say P is a k-polytope. A subset $F \subseteq P$ forms a *face* if there exists a hyperplane H such that $H \cap P = F$ and $P \setminus F$ lies in exactly one of the half spaces formed by H. In this case, we say that H is a *support hyperplane* of P. A face of a k-polytope P whose affine dimension is $k-1$ is a *facet*. A k-polytope all of whose facets contain k vertices is said to be *simplicial*. We will say that two polytopes, $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$, are *combinatorially equivalent*, denoted $P \sim Q$, if there exists a bijection $\pi : \mathbb{R}^d \to \mathbb{R}^e$ between their vertices preserving inclusion of faces. Finally, a function $f \colon \mathbb{R}^d \to \mathbb{R}^e$ such that $f(x) = a + L(x)$ for $a \in \mathbb{R}$ and some linear function $L \colon \mathbb{R}^d \to \mathbb{R}^e$ is an *affine function*. Two polytopes $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ are *isomorphic*, denoted $P \approx Q$, if there is an affine function $A \colon \mathbb{R}^d \to \mathbb{R}^e$, injective when restricted to the affine span of P, such that $A(P) = Q$. Isomorphic polytopes are combinatorially equivalent, but the opposite is not necessarily true.

A classic example, and one that will enter into our discussion, is the *cyclic polytope*. Define the mapping $x \colon \mathbb{R} \to \mathbb{R}^d$ by $x(t) = (t, t^2, t^3, \dots, t^d)$. The image of x is called the *moment curve*, and the cyclic d-polytope with n vertices, denoted $C_d(n)$, is $\operatorname{conv}(x(t_i))$ with $t_1 < t_2 < \cdots < t_n$. It is worth noting that the specific value of each t_i is unimportant. We require only that they are distinct and ordered. Their structure is well known; they are simplicial polytopes with $\binom{n-\lfloor \frac{d+1}{2} \rfloor}{n-d} + \binom{n-\lfloor \frac{d+2}{2} \rfloor}{n-d}$ facets [3]. We will, however, show a classic result of the defining characteristic of cyclic polytopes, first noted by Gale [3]. The proof we will follow is due to Ziegler [8].

Theorem 1. *Gale's Evenness Condition: Let $n > d \geq 2$. The cyclic polytope $C_d(n) = conv(x(t_1), \dots, x(t_n))$ is a simplicial d-polytope such that for a d-subset $S = \{i_1, \dots i_d\} \subset \{1, \dots, n\}$, the set of vertices $D = \{x(t_{i_1}), \dots x(t_{i_d})\}$ forms a facet of $C_d(n)$ if and only if, for all l, k in the complement of S in $\{1, \dots, n\}$ with $(k < l)$, there is an even number of i_j's in S such that $k < i_j < l$.*

Proof. Looking at the Vandermonde determinant, note that:

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x(t_0) & x(t_1) & \cdots & x(t_d) \end{pmatrix} = \prod_{0 \leq i < j \leq d} (t_j - t_i).$$

Since this vanishes only when $t_i = t_j$ for some $i \neq j$, we see that no $d+1$ vertices of $C_d(n)$ are affinely dependent, which means that any facet contains only d vertices, so $C_d(n)$ is simplicial.

Now, define

$$F_S(x) := \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x & x(t_{i_1}) & \cdots & x(t_{i_d}) \end{pmatrix} \quad \text{for } x = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}$$

and $\{i_1, \ldots, i_d\} = S$ as above. $F_S(x) = 0$ for x lying on the hyperplane intersecting $x(t_{i_1}), \ldots, x(t_{i_d})$. Note that $F_S(x(t))$ is a polynomial of degree d which vanishes at $t = t_i$ for $i \in S$ and changes sign at each such zero. The set $\{x(t_{i_1}), \ldots x(t_{i_d})\}$ forms a facet of $C_d(n)$ if and only if $F_S(x(t))$ has the same sign at all $k \in S^c$. So $F_S(x(t))$ must have an even number of sign changes between $t = t_k$ and $t = t_l$ with $k, l \in S^c$ and $k < l$, and hence an even number of elements $i_j \in S$ with $k < i_j < l$. $\square$

0.3 The Problem

Returning then to the subject of representations, let ρ be an arbitrary real-valued representation of $\mathbb{Z}/n\mathbb{Z}$ for some n with $\rho(a)$ as given in the matrix (4). Thinking of ρ as a function from $\mathbb{Z}/n\mathbb{Z}$ to $\mathbb{R}^{d \times d}$, we can "flatten" the image of $\rho(a)$ and then reorder the entries to get $\rho(a) = (b_1(a), \ldots, b_m(a), 0, \ldots, 0) \in \mathbb{R}^{d^2}$ where, for each $b_i(a)$, either

$$\begin{aligned} b_i(a) &= (1) \\ b_i(a) &= ((-1)^a) \quad \text{or} \\ b_i(a) &= \left(\cos(\frac{2\pi ka}{n}), -\sin(\frac{2\pi ka}{n}), \sin(\frac{2\pi ka}{n}), \cos(\frac{2\pi ka}{n})\right), \end{aligned}$$

depending on the character(s) that the corresponding B_i is derived from in (4). For the set $\{\rho(a) \mid a \in \mathbb{Z}/n\mathbb{Z}\}$ then, we can obviously map each point bijectively to $\mathbb{R}^e$ for some $e < d^2$ such that the image is the set $\{\rho'(a) = (b_1(a), \ldots, b_m(a)) \mid a \in \mathbb{Z}/n\mathbb{Z}\}$. The main goal of the thesis is to describe the face structure of

$$P = \text{conv}\{\rho(a) \mid a \in \mathbb{Z}/n\mathbb{Z}\} \approx \text{conv}\{(b_1(a), \ldots, b_m(a)) \mid a \in \mathbb{Z}/n\mathbb{Z}\}.$$

To simplify the problem somewhat more, let $b_i(a) = 1$ for some i, where $b_i(a)$ was derived from the identity character $\chi_0(a)$. For some j then, the j^{th} coordinate of $\rho'(a)$ is 1 for all a. Once again, there is a bijection f defined on the image of ρ' such that

$$f(x_1, \ldots, x_{j-1}, 1, x_{j+1} \ldots, x_e) = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_e) \in \mathbb{R}^{e-1},$$

and hence the polytope in $\mathbb{R}^{e-1}$ defined as $\text{conv}(f(\rho'(a)))$ is isomorphic to the polytope defined by $\text{conv}(\rho'(a)) \subset \mathbb{R}^e$.

Now, let P be $\text{conv}(\{\rho'(a) \mid a \in \mathbb{Z}/n\mathbb{Z}\}) \subset \mathbb{R}^e$ for $\rho'(a)$ as above. Further, assume that for all $a \in \mathbb{Z}/n\mathbb{Z}$, the j^{th} coordinate of $\rho'(a)$is equal to a fixed scalar multiple of the i^{th} coordinate: so $\rho'(a) = (x_1, \ldots, x_i, \ldots, x_{j-1}, c \cdot x_i, x_{j+1}, \ldots, x_e)$ for a scalar c. Once again, there is an obvious bijection $f\colon \mathbb{R}^e \to \mathbb{R}^{e-1}$ such that $f(x_1, \ldots, x_e) = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_e)$. From this fact, we gain two results. First, we see that we can reduce

$$b_i(a) = (\cos(\frac{2\pi ka}{n}), -\sin(\frac{2\pi ka}{n}), \sin(\frac{2\pi ka}{n}), \cos(\frac{2\pi ka}{n}))$$

$$\text{to} \quad (\cos(\frac{2\pi ka}{n}), \sin(\frac{2\pi ka}{n}))$$

by applying the argument to both the second and fourth entries. Secondly, this means that if any b_i appears more than once in ρ', we can remove all but one of them without changing the isomorphism class of the polytope defined as $\operatorname{conv}(\{\rho'(a) \mid a \in \mathbb{Z}/n\mathbb{Z}\})$.

As an example, let $n = 6$. For $a \in \mathbb{Z}/6\mathbb{Z}$, let

$$\rho(a) = \begin{pmatrix} 1^a & 0 & 0 & 0 \\ 0 & (-1)^a & 0 & 0 \\ 0 & 0 & \cos(\frac{2\pi a}{6}) & -\sin(\frac{2\pi a}{6}) \\ 0 & 0 & \sin(\frac{2\pi a}{6}) & \cos(\frac{2\pi a}{6}) \end{pmatrix}.$$

From above, we view this as a mapping into $\mathbb{R}^{4\times 4}$, and "flatten" the image of ρ so that we have

$$a \mapsto \left(1^a, (-1)^a, \cos(\frac{2\pi a}{6}), -\sin(\frac{2\pi a}{6}), \sin(\frac{2\pi a}{6}), \cos(\frac{2\pi a}{6}), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)\right).$$

The resulting polytope for this representation is the convex hull of the set Y of points of this form in $\mathbb{R}^{4^2}$ for $a = \{0, 1, \ldots, 5\}$. By the above reasoning, we can map the image of ρ to $\mathbb{R}^3$ and study instead the 3-polytope defined as $\operatorname{conv}(X)$ where $X = \{((-1)^a, \cos(\frac{2\pi a}{6}), \sin(\frac{2\pi a}{6})) \mid a \in \mathbb{Z}/6\mathbb{Z}\}$. To see this, define a function

$$\pi\colon \mathbb{R}^{16} \longrightarrow \mathbb{R}^3$$

$$(x_1, \ldots, x_{16}) \mapsto (x_2, x_3, x_5),$$

which is clearly injective on Y and therefore gives an isomorphism of polytopes, $\operatorname{conv}(X) \approx \operatorname{conv}(Y)$.

For any arbitrary representation ρ, we have now shown 1) that the trivial character χ_0 can be ignored when studying the facial structure of the convex hull of the image of ρ, and 2) that thinking of ρ as a direct sum of characters, we may assume that each character in the sum is unique, i.e., has multiplicity one.

As a result of the above observations, we can confine our attention to the set $\operatorname{conv}(X)$ where X is the image of any representation ρ such that, for a given n,

$$\rho = \chi_{i_1} \oplus \chi_{-i_1}, \oplus \cdots \oplus \chi_{i_k} \oplus \chi_{-i_k}$$

where $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$ and all i_j's are distinct[1]. As a note, we will use $\rho(a)$ to denote the image of the representation and ρ to denote the representation itself.

For each representation $\chi_i \oplus \chi_{-i}$ (or resp. $\chi_{\frac{n}{2}}$ if n is even), call this V_i (resp. $V_{\frac{n}{2}}$). Then the complete set of real-valued representations of $\mathbb{Z}/n\mathbb{Z}$, which are direct sums of the V_i's, forms a lattice ordered by inclusion, which we will call $\mathcal{L}$.

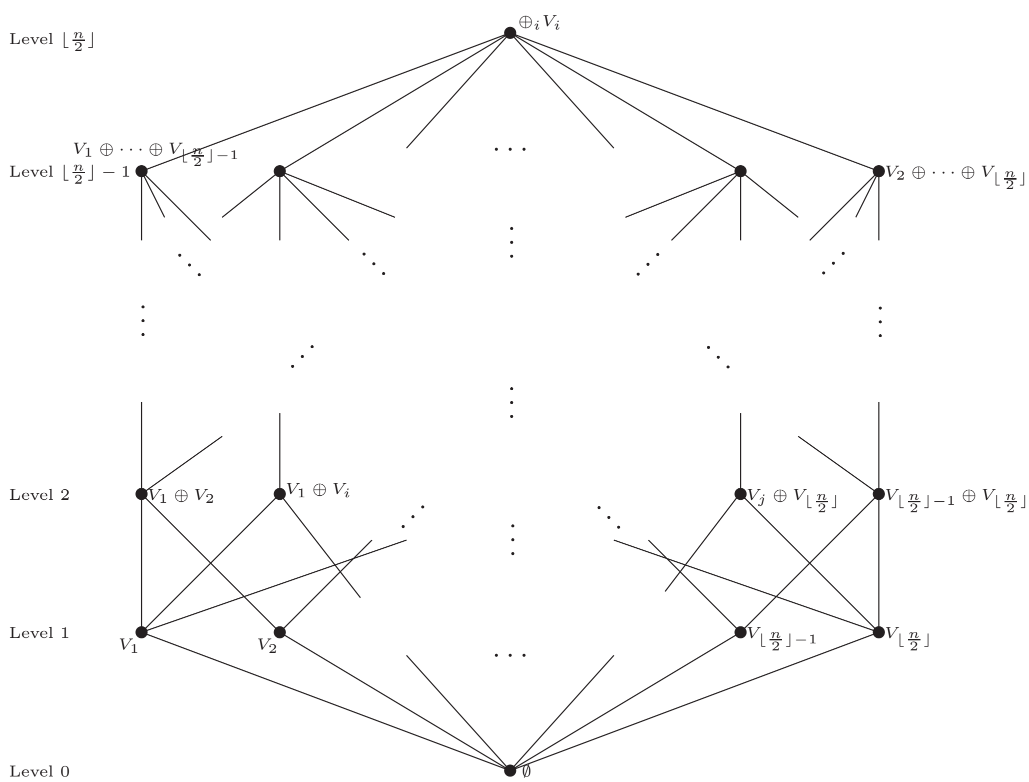

Figure 1: The representation lattice for $\mathbb{Z}/n\mathbb{Z}$

The lattice corresponding to the complete set of representations of $\mathbb{Z}/n\mathbb{Z}$ for any fixed n will be denoted $\mathcal{L}_n$.

We note first that the polytopes corresponding to representations from level 0 are 0-dimensional, where all n vertices share a single point. Those from level $\lfloor \frac{n}{2} \rfloor$ are $(n-1)$-dimensional simplices with n vertices by Theorem 2 below. Polytopes derived from level 1 are regular k-gons where, for V_j, we have $k = \frac{n}{(j,n)}$, and each vertex has multiplicity (j, n).

The extent of present investigations concerning the remainder of the polytopes derived from the representations of the lattice is mainly confined to two areas, and our problem can be seen as a generalization of both of these.

The first is the case where a representation of $\mathbb{Z}/n\mathbb{Z}$ is of the form $\rho(a) =$

$$\left(\cos(\tfrac{1(2\pi a)}{n}), \sin(\tfrac{1(2\pi a)}{n}), \cos(\tfrac{2(2\pi a)}{n}), \sin(\tfrac{2(2\pi a)}{n}), \ldots, \cos(\tfrac{m(2\pi a)}{n}), \sin(\tfrac{m(2\pi a)}{n})\right)$$

[1]Note, however, that if n is even and the character $\chi_{\frac{n}{2}}$ appears in ρ, then of course $\chi_{-\frac{n}{2}} (= \chi_{\frac{n}{2}})$ does not.

for $a \in \mathbb{Z}/n\mathbb{Z}$. For such ρ, the convex hull of its image is combinatorially equivalent to the $2m$-dimensional[2] cyclic polytope of n vertices, and is referred to as the *trigonometric cyclic polytope.*

Theorem 2. *For such a representation ρ, the polytope P defined by its convex hull is cyclic.*

Proof. Again, we will study the determinant and use the identity:

$$\det \begin{pmatrix} 1 & 1 & \dots & 1 \\ \cos(\theta_0) & \cos(\theta_1) & \dots & \cos(\theta_{2m}) \\ \sin(\theta_0) & \sin(\theta_1) & \dots & \sin(\theta_{2m}) \\ \vdots & \vdots & \ddots & \vdots \\ \cos(m\theta_0) & \cos(m\theta_1) & \dots & \cos(m\theta_{2m}) \\ \sin(m\theta_0) & \sin(m\theta_1) & \dots & \sin(m\theta_{2m}) \end{pmatrix} = 4^{n^2} \prod_{0 \le i < j \le 2m} \sin \frac{1}{2}(\theta_j - \theta_i)$$

for $\theta_i = \frac{2\pi a_i}{n}$ and $a_i \in \mathbb{Z}/n\mathbb{Z}$. As above, this product vanishes only when $\theta_i = \theta_j$ for some $i \neq j$, so we see that no $2m$ vertices are affinely dependent, meaning that P is simplicial, once again. From here we continue as in the previous theorem and the result follows. $\square$

Note that for any n, at least one polytope from this combinatorial equivalence class will exist on each level of $\mathcal{L}_n$: for any level k of $\mathcal{L}_n$, it is the representation $\rho = V_1 \oplus V_2 \oplus \cdots \oplus V_{k-1} \oplus V_k$.

The other set of polytopes in $\mathcal{L}$ which are studied fairly well are all those in level 2, which Smilansky refers to as the *bi-cyclic polytopes* [6],[7]. We will now state briefly his results.

Smilansky defines a mapping into the torus imbedded in 4-space:

$$\eta \colon \mathbb{R}^2 \to T \subseteq \mathbb{R}^4$$

such that

$$\eta(x, y) = \left(\cos(\frac{2\pi}{n}x), \sin(\frac{2\pi}{n}x), \cos(\frac{2\pi}{n}y), \sin(\frac{2\pi}{n}y) \right).$$

Consider the lattice $\Theta = n\mathbb{Z} \times n\mathbb{Z} \subseteq \mathbb{R}^2$ and for $p, q \in \{1, \dots, \lfloor \frac{n-1}{2} \rfloor\}$, define the set $S_{p,q} = \{a(p, q) \mid a \in \mathbb{Z}/n\mathbb{Z}\}$. Then $\Lambda_{p,q} = S_{p,q} + \Theta$ is also a lattice such that $\mathrm{conv}(\eta(\Lambda_{p,q}))$ is the polytope whose representation $\rho = V_p \oplus V_q$ exists on level 2 of $\mathcal{L}_n$.

Smilansky showed that for any $\alpha_1, \alpha_2, \alpha_3, \alpha_4, d \in \mathbb{R}$ with all α_i non-zero, and the hyperplane

$$H := \alpha_1 \cos(\frac{2\pi}{n}x) + \alpha_2 \sin(\frac{2\pi}{n}x) + \alpha_3 \cos(\frac{2\pi}{n}y) + \alpha_4 \sin(\frac{2\pi}{n}y) - d,$$

$\eta^{-1}(H)$ is a level set in $\mathbb{R}^2$ resembling an ellipse or a vertical or horizontal line. If $d = 1$ and $\alpha_1, \dots, \alpha_4$ are such that H is a support hyperplane intersecting P on a

[2] If n is even and $m = \frac{n}{2}$, then P is a $(2m-1)$-dimensional cyclic polytope of n vertices and is also a $(n-1)$-dimensional simplex corresponding to the maximal representation.

facet, then the set of vertices of P on the facet are mapped to points in $\Lambda_{p,q}$ in one of two ways. In some fundamental region (an $n \times n$ square) of $\Lambda_{p,q}$, the points will either form: 1) a parallelogram of area n such that two of its edges have positive slope and the other pair has negative slope or 2) a closed vertical (resp. horizontal) strip in the plane bounded by distinct vertical (resp. horizontal) lines, each intersecting at least two such points.

As an example, let P by the 4-polytope defined by conv(A) where

$$A = \{(\cos(\frac{2\pi 2a}{7}), \sin(\frac{2\pi 2a}{7}), \cos(\frac{2\pi 3a}{7}), \sin(\frac{2\pi 3a}{7})) \mid a \in \mathbb{Z}/7\mathbb{Z}\}.$$

Then the image of $\eta^{-1}(A)$ is a lattice generated by $(2a, 3a)$:

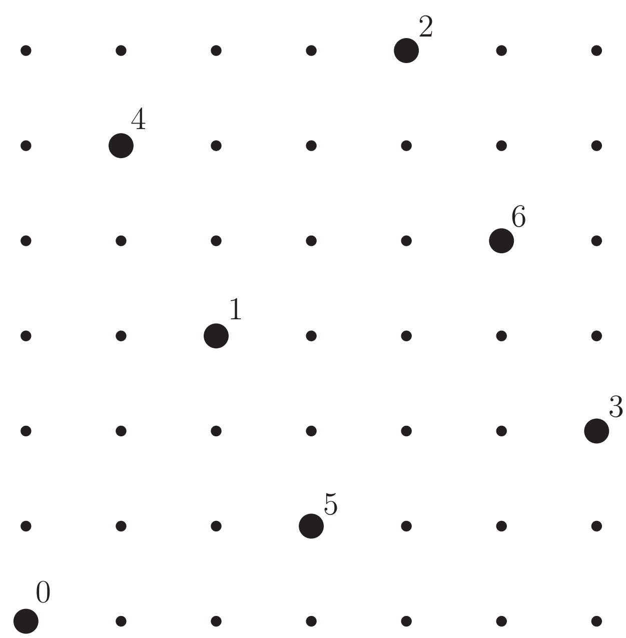

Figure 2: $\eta^{-1}(A)$

We find, then, that the faces of P correspond to the parallelograms of this lattice with vertices $(1, 2, 6, 5), (1, 6, 3, 5), (0, 1, 5, 4)$, etc.

Our polytope representations are a natural generalization of both the cyclic and bi-cyclic polytopes, and we will refer to them as "multi-cyclic." The goal of the remainder of this paper is to describe the representations in the remaining levels of the lattice and some relations between them.

Chapter 1

Generalized Vandermonde Matrices

We have seen that the multi-cyclic polytopes are a natural generalization of the trigonometric cyclic polytope described earlier (which we have already seen to be combinatorially equivalent to the cyclic polytopes). Naturally, one would wonder whether there is a corresponding generalization of the 'classical' cyclic polytopes. We will see that by imbedding the images of our group representations into $\mathbb{C}^m$ for some m, we do in fact get a set of vertices which have a form similar to such a generalization. The matrix representations of the vertex sets have the form of *generalized Vandermonde determinants* (cf.[1]).

1.1 Generalized Vandermonde Determinants

For $X = \{x_1, \ldots, x_n\}$, let $\mathrm{VDM}_n(X)$ denote the familiar Vandermonde matrix:

$$VDM_n(X) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}$$

Let the columns of a matrix M be defined as $(x_i^{\mu_1}, x_i^{\mu_2}, \ldots, x_i^{\mu_n}) \in \mathbb{R}^n$ where the x_i's are distinct real numbers and the μ_i's are non-negative ordered integers $0 \leq \mu_1 < \mu_2 < \cdots < \mu_n$. Then

$$M(X) = \begin{pmatrix} x_1^{\mu_1} & x_2^{\mu_1} & \cdots & x_n^{\mu_1} \\ x_1^{\mu_2} & x_2^{\mu_2} & \cdots & x_n^{\mu_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\mu_n} & x_2^{\mu_n} & \cdots & x_n^{\mu_n} \end{pmatrix}$$

is a *generalized Vandermonde matrix*.

The determinants of these matrices, like those of Vandermonde matrices, are based on polynomials of n variables where M is a $n \times n$ matrix. It turns out that

$$\det(M(X)) = \det(VDM_n(X)) \cdot S_\lambda \quad [1],$$

where S_λ is a *Schur function*, a symmetric function of $(x_1, \ldots, x_n)$ consisting of monomials. Schur functions will be discussed in the next section.

1.2 Schur Functions

A *Young diagram* is a set of boxes in left-justified rows, with a weakly decreasing number of boxes in each row. Say the total number of boxes for such a diagram is 'n.' Then the Young diagram can be seen to represent a partition of $[n] := \{1, 2, \ldots, n\}$ into m subsets, where m is the number of rows in the diagram. In this case we will say then that the partition has length m. A *filling* for a given Young diagram is a manner of placing a (not necessarily distinct) positive integer in each box. A *Young tableaux* is a filling that is weakly increasing across each row and strictly increasing down each column. An example of this is displayed in figure 1.1.

1	1	3	4	4
2	3	4		
4	4	5		
5	6			

Figure 1.1: A Young tableaux

For a partition λ of length m corresponding to a Young diagram and its m rows, there is associated a Schur function, which can be determined from a given diagram.

Let λ be a Young diagram and let T represent an arbitrary filling of λ. Then let x^T denote the monomial $x_1^{i_1} \cdot x_2^{i_2} \cdots x_n^{i_n}$ where x_j is a variable and the exponents i_j denote the number of times the integer j appears in the filling T of λ. As an example, the monomial for the diagram above is $x_1^2 \cdot x_2 \cdot x_3^2 \cdot x_4^5 \cdot x_5^2 \cdot x_6$. The Schur function for the diagram λ, then, is the sum of all monomials $S_\lambda(x_1, \ldots, x_m) = \sum x^T$ for all possible fillings T of λ with a fixed set of m integers.

1.3 Multi-cyclic Polytopes as Generalized Vandermonde Matrices

To see the connection to our problem, let us begin by operating on the vertex matrix of an arbitrary multi-cyclic polytope P:

$$V = \begin{pmatrix} \cos(\theta_0 k_1) & \cos(\theta_1 k_1) & \cdots & \cos(\theta_{n-1} k_1) \\ \sin(\theta_0 k_1) & \sin(\theta_1 k_1) & \cdots & \sin(\theta_{n-1} k_1) \\ \vdots & \vdots & \ddots & \vdots \\ \cos(\theta_0 k_m) & \cos(\theta_1 k_m) & \cdots & \cos(\theta_{n-1} k_m) \\ \sin(\theta_0 k_m) & \sin(\theta_1 k_m) & \cdots & \sin(\theta_{n-1} k_m) \end{pmatrix}$$

where $\theta_a = \frac{2\pi a}{n}$ for $a \in \mathbb{Z}/n\mathbb{Z}$[1].

Replacing $\cos(x)$ and $\sin(x)$ respectively by $\frac{e^{ix}+e^{-ix}}{2}$ and $\frac{e^{ix}-e^{-ix}}{2i}$, we use row operations to transform V into:

$$(\frac{-1}{2i})^m \begin{pmatrix} e^{i\theta_0 k_1} & \cdots & e^{i\theta_{n-1} k_1} \\ e^{-i\theta_0 k_1} & \cdots & e^{-i\theta_{n-1} k_1} \\ \vdots & \ddots & \vdots \\ e^{i\theta_0 k_m} & \cdots & e^{i\theta_{n-1} k_m} \\ e^{-i\theta_0 k_m} & \cdots & e^{-i\theta_{n-1} k_m} \end{pmatrix}$$

Letting $\omega_a = e^{\frac{2\pi i \cdot a}{n}}$, we then get

$$(\frac{-1}{2i})^m \begin{pmatrix} \omega_0^{k_1} & \cdots & \omega_{n-1}^{k_1} \\ \omega_0^{-k_1} & \cdots & \omega_{n-1}^{-k_1} \\ \vdots & \ddots & \vdots \\ \omega_0^{k_m} & \cdots & \omega_{n-1}^{k_m} \\ \omega_0^{-k_m} & \cdots & \omega_{n-1}^{-k_m} \end{pmatrix}$$

where k is a positive integer and '$-k$' is to be interpreted modulo n. After exchanging rows so that the exponents appear in increasing order, we clearly have a generalized Vandermonde matrix in the variables ω_a.

For $X = \{x_1, x_2, \ldots, x_n\}$ (with distinct x_i's) and integers $\mu_{n-1} \geq \mu_{n-2} \geq \cdots \geq \mu_1 \geq 0$, let $\mu = (\mu_{n-1}, \mu_{n-2}, \ldots, \mu_1, 0)$ be a partition of length n where each integer μ_i corresponds to the exponent in the i^{th} row of a generalized Vandermonde matrix $M(X)$. Now, let $\delta = (n-1, n-2, \ldots, 1, 0)$ be a partition corresponding to the exponents for the rows of the Vandermonde matrix $\mathrm{VDM}_n(X)$. Then for $\det(M(X)) = \det(\mathrm{VDM}_n(X)) \cdot \mathrm{S}_\lambda(X)$ we have that λ is the partition $\mu - \delta$ and hence $S_\lambda(X)$ is the sum of monomials corresponding to fillings of λ with $x_1, \ldots, x_n$. Following the example of Gale then, we can study the behavior of these determinants and in turn study the polytope that such a matrix $M(X)$ represents.

[1]We must keep in mind that in the case that n is even and $\chi_{\frac{n}{2}}$ is added to the representation determining P, then the $2d+1^{th}$ row of V will be given by $\begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix}$.

Define the determinant function

$$V(x) = \left(\frac{-1}{2i}\right)^m \det \begin{pmatrix} 1 & \cdots & 1 & 1 \\ \omega_{j_1}^{k_1} & \cdots & \omega_{j_d}^{k_1} & \omega_x^{k_1} \\ \omega_{j_1}^{k_2} & \cdots & \omega_{j_d}^{k_2} & \omega_x^{k_2} \\ \vdots & \ddots & \vdots & \vdots \\ \omega_{j_1}^{-k_2} & \cdots & \omega_{j_d}^{-k_2} & \omega_x^{-k_2} \\ \omega_{j_1}^{-k_1} & \cdots & \omega_{j_d}^{-k_1} & \omega_x^{-k_1} \end{pmatrix}$$

with ω_{j_i} as defined above. The vertices corresponding to $\{j_1, \dots, j_d\}$ will be contained in a facet of P if and only if $V(x)$ has the same sign for all $j \in S^c$. If $V(j') = 0$ for some $j' \in S^c$, then we add j' to the list of fixed test points and continue evaluating the sign of $V(x)$ for the remaining $j \in S^c$. (This is the same tool that we used in our determinantal approach to cyclic polytopes in the introduction.)

We note that for $X = (\omega_{j_1}, \dots, \omega_{j_d}, \omega_x)$,

$$V(x) = \det(\mathrm{VDM}_{d+1}) \cdot \mathrm{S}_\lambda = \det(\mathrm{VMD}_{d+1}) \cdot (\prod_{i=1}^{d}(\omega_x - \omega_{j_i}))\mathrm{P}_M(X) \quad [1].$$

where $\mathrm{P}_M(X)$ is a polynomial in $(\omega_{j_1}, \dots, \omega_{j_d}, \omega_x)$.

Though connecting the problem to the determinants of these matrices and to Schur functions is interesting, we have only introduced a new tool with which to compute the face structure of any given polytope, and the general problem at this point is still open.

Chapter 2

Duality Theorem of Representations

2.1 Gale Diagrams

One method of visualizing and studying higher-dimensional polytopes is through the use of *Gale Diagrams.* Though they are generally only applied to polytopes having 'few' vertices (a d-polytope with $< d+4$ vertices), we shall see that for our purposes, the underlying idea is one that will characterize our entire family of polytopes. Our exposition of Gale diagrams will follow [8].

Let $X = \{x_1, x_2, \ldots, x_n\}$ with $x_i \in \mathbb{R}^d$ be the columns[1] of a $d \times n$ matrix X.[2]

Definition. The *affine dependencies* of X are:

$$\mathrm{Dep}(X) = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \mid X \cdot \lambda = 0, \textstyle\sum_{i=1}^n \lambda_i = 0\}.$$

If we define $\tilde{X}$ as $(\tilde{x}_1, \ldots, \tilde{x}_n)$ for $\tilde{x}_i = \begin{bmatrix} 1 \\ x_i \end{bmatrix} \in \mathbb{R}^{d+1}$ then $\mathrm{Dep}(\tilde{X}) = \ker(\tilde{X})$.

Definition. We define the *signed vectors* of X as

$$\mathbb{V}(X) = \{\mathrm{sign}(\lambda) \mid \lambda \in \mathrm{Dep}(X)\}$$

where, for $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$,

$$\mathrm{sign}(\lambda) = (\mathrm{sign}(\lambda_1), \ldots, \mathrm{sign}(\lambda_n))$$

and, for $\lambda_i \in \mathbb{R}$,

$$\mathrm{sign}(\lambda_i) = \begin{cases} - & \text{if} \quad \lambda_i < 0 \\ 0 & \text{if} \quad \lambda_i = 0 \\ + & \text{if} \quad \lambda_i > 0 \end{cases} .$$

We order the signed vectors component-wise as well where $0 < -$, $0 < +$, and $+, -$ are incomparable. In this way we can define:

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[1] As a convention, we will identify elements of $\mathbb{R}^n$ as column matrices, and we will denote row vectors by x^{tr} for $x \in \mathbb{R}^n$.

[2] For such sets, we will assume from now on that $\{x_1, \ldots, x_n\}$ spans $\mathbb{R}^d$.

Definition. The *circuits* of X are the minimal (non-zero) signed vectors:

$$\mathbf{C}(X) = \{v \in \mathbb{V}(X) \mid w \leq v \Rightarrow v = w\}.$$

Example: Let $\tilde{W} = \{\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{w}_4\}$ where $\tilde{w}_i = \binom{1}{w_i}$ and w_i is a vertex of the unit square:

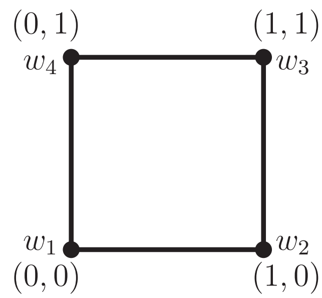

Figure 2.1: A square

Then

$$\mathrm{Dep}(\tilde{W}) = \ker \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \mathrm{span} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$$

Then clearly,

$$\mathbb{V}(\tilde{W}) = \{(+,-,+,-), (-,+,-,+), (0,0,0,0)\}$$

and

$$\mathbf{C}(\tilde{W}) = \{(+,-,+,-), (-,+,-,+)\}.$$

Definition. Finally, define the *value vectors* of our set X as

$$\mathrm{Val}(X) = \{\mathbf{c}^{tr}\mathbf{X} \mid \mathbf{c} \in (\mathbb{R}^d)\} = \mathrm{im}(X^{tr})$$

Now, let $\tilde{X}$ be as defined above, and let f be a function $f \colon \mathbb{R}^{d+1} \to \mathbb{R}$ such that $f(x) = \tilde{\mathbf{c}}^{tr} \cdot \mathbf{x}$ for some $\tilde{\mathbf{c}} = \binom{c_0}{c} \in \mathbb{R}^{d+1}$. The set $\{x \in \mathbb{R}^{d+1} \mid f(x) = 0\}$ defines a hyperplane in $\mathbb{R}^{d+1}$. In this case, we have

$$\mathrm{Val}(\tilde{X}) = \{\tilde{\mathbf{c}}^{tr}\tilde{\mathbf{X}} \mid \tilde{\mathbf{c}} \in (\mathbb{R}^{d+1})\} = \{c_0 + \mathbf{c}^{tr}\tilde{x}_i \mid c_0 \in \mathbb{R}, \mathbf{c} \in \mathbb{R}^d, \tilde{x}_i \in \tilde{X}\}.$$

$\mathrm{Val}(\tilde{X})$ is, geometrically, the oriented distance of each $\tilde{x}_i \in \tilde{X}$ from the defined hyperplane. As above, then, we will define the *signed covectors* as

$$\mathbb{V}^*(X) = \{\mathrm{sign}(\mathbf{c}^{tr}\mathbf{X}) \mid \mathbf{c} \in \mathbb{R}^d\}.$$

Then, we have the *signed cocircuits* of X, denoted $\mathbf{C}^*(X)$, as we had in the linear case; that is, $\mathbf{C}^*(X)$ is the set of minimal (non-zero) signed covectors of X.

As we will see, any one of the 1) signed vectors, 2) circuits, 3) signed covectors, and 4) signed cocircuits determines the other three. However, it is rather easy to read off the faces of $\mathrm{conv}(X)$ directly from $\mathbf{C}^*(X)$. Any support hyperplane $H \subseteq \mathbb{R}^d$ will correspond to a signed cocircuit of X such that all non-zero entries have the same sign. The zero entries will correspond to the set of $x_i \in X$ lying on H, and hence on the face.

We now prove a duality between $\mathrm{Val}(X)$ and $\mathrm{Dep}(X)$:

Theorem 3. *For a set X of n vectors spanning $\mathbb{R}^d$, (a) $\mathrm{Val}(X) = (\mathrm{Dep(X)})^\perp$ and (b) $\mathrm{Dep}(X) = (\mathrm{Val(X)})^\perp$*

Proof. (a): First, note that $\dim(\mathrm{Dep}(X)) = n - d$. Then for the orthogonal complement in $\mathbb{R}^n$, $\dim((\mathrm{Dep}(X))^\perp) = n - (n - d) = d$. Now, $\dim(\mathrm{Val}(X)) = \dim(\mathrm{im}(X^{tr})) = \mathrm{rank}(X^{tr}) = \mathrm{rank}(X) = n - \dim(\ker(X)) = n - (n - d) = d$.

Then, $(\mathrm{Dep}(X))^\perp = \{u \in \mathbb{R}^n \mid u \cdot v = 0 \ \forall v \in \mathrm{Dep}(X)\}$. Then for $cV \in \mathrm{Val}(X)$ and $v \in \mathrm{Dep}(X)$, $(cV)v = c(Vv) = c\mathbf{0} = 0$.
(b): $((\mathrm{Dep}(X))^\perp)^\perp = \mathrm{Dep}(X)$. Result from (a) applies. □

Theorem 4. *For some X as above, a duality exists between the circuits $\mathbf{C}$ and cocircuits $\mathbf{C}^*$:*

$$\mathbf{C}(X) = \mathbf{C}^*(\mathrm{Dep}(X)), \qquad \mathbf{C}^*(X) = \mathbf{C}(\mathrm{Dep}(X))$$

Proof. See [4]. □

Taking, again, our vertex points $X = \{x_1, \ldots, x_n\}$ to be the columns of a $d \times n$ matrix, the Gale diagram is found as follows: Given such X, we first linearize it, getting the matrix $\tilde{X} \subseteq \mathbb{R}^{(d+1)\times n}$ with column vectors

$$\tilde{x}_i = \begin{pmatrix} 1 \\ x_i \end{pmatrix}$$

for $x_i \in X$.

Then there is a matrix $G \in \mathbb{R}^{n\times(n-(d+1))}$ such that

$$\mathrm{Dep}(\tilde{X}) = \{Gv \mid v \in \mathbb{R}^{n-(d+1)}\}.$$

Since $\mathrm{Dep}(\tilde{X}) = \ker(\tilde{X})$, we know that the columns of G are exactly a basis for $\ker(\tilde{X})$. Here $n - (d+1) = \dim(\mathrm{Dep}(\tilde{X}))$. Denote the *rows* of G by $g_1, \ldots, g_n$. The set $\{g_1^{tr}, \ldots, g_n^{tr}\}$ with $g_i^{tr} \in \mathbb{R}^{n-(d+1)}$ is the Gale diagram for X, and it is unique up to linear change of coordinates. Using this method, we can then read off the circuits of the Gale diagram of X to determine the face structure of $\mathrm{conv}(X)$.

Example: The Gale diagram for figure (2.1) is

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \qquad \text{given by figure 2.2.}$$

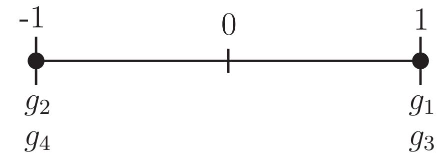

Figure 2.2: Gale diagram for the unit square

Then $\mathrm{Val}(G) = \{c \cdot (1, -1, 1, -1) \mid c \in \mathbb{R}\} = \{(c, -c, c, -c) \mid c \in \mathbb{R}\}$. So $\mathbf{C}^*(G) = \pm(+, -, +, -) = \mathbf{C}(X)$, as expected.

Example: We shall look at level $\lfloor \frac{n}{2} \rfloor - 1$ of the representation lattice $\mathcal{L}_{15}$. For some $j \in \{1, \ldots, 7\}$, we have

$$\rho(a) = \left(\cos(\frac{2\pi a}{15}), \sin(\frac{2\pi a}{15}), \ldots, \widehat{\cos(\frac{2\pi j a}{15})}, \widehat{\sin(\frac{2\pi j a}{15})}, \ldots, \cos(\frac{2\pi 7a}{15}), \sin(\frac{2\pi 7a}{15}) \right)$$

. For the example, let $j = 4$. Then $X = (x_0, \ldots, x_{14})$ where

$$x_a = \begin{pmatrix} \cos(\frac{2\pi a}{15}) \\ \sin(\frac{2\pi a}{15}) \\ \vdots \\ \cos(\frac{6\pi a}{15}) \\ \sin(\frac{6\pi a}{15}) \\ \cos(\frac{10\pi a}{15}) \\ \sin(\frac{10\pi a}{15}) \\ \vdots \\ \cos(\frac{14\pi a}{15}) \\ \sin(\frac{14\pi a}{15}) \end{pmatrix}$$

and the polytope $P = \mathrm{conv}(X)$. Then by direct calculation, $\ker(\tilde{X})$ is the span of the columns of

$$G = \begin{pmatrix} g_0 \\ \vdots \\ g_{14} \end{pmatrix}$$

where $g_a = \left(\cos(\frac{8\pi a}{15}), \sin(\frac{8\pi a}{15})\right)$, and $\tilde{X}$ is the linearization of X as before. Since 15 and 4 are relatively prime, one sees easily that $\mathrm{conv}(\ker(\tilde{X}))$ is a regular 15-gon

Since the circuits $\mathbf{C}(G)$ of G are dual to the cocircuits $\mathbf{C}^*(\tilde{X})$ of $\tilde{X}$, we want to find minimal circuits of G, i.e., $\lambda = (\lambda_0, \ldots, \lambda_{14})$ of minimal support such that $\sum \lambda_i g_i = 0$, with not all λ_i equal to zero. For each such circuit, the set of g_i's with zero coefficients corresponds to a facet of the original polytope P if all non-zero λ_i's

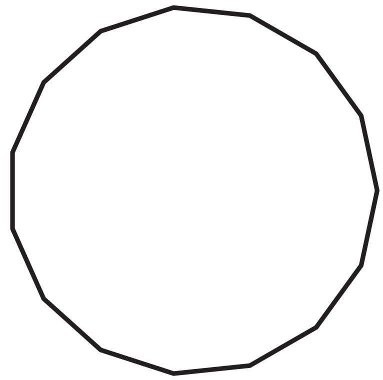

Figure 2.3: A 15-gon

are of the same sign. When we have such a circuit, again, this corresponds to a cocircuit of V, and hence a hyperplane supporting it, meaning that the set of all vertices not spanning the hyperplane all lie on one "side" of it (hence all having the same sign). Since P is a 12-polytope with 15 vertices, P is not simplicial if there is a facet containing more than 12 vertices. This in turn corresponds to a circuit of G with only 1 or 2 g_i's having nonzero coefficients. Since conv(G) is a regular polygon centered at the origin, there is clearly not such a circuit with only one such g_i. For there to be a circuit with two such g_i's, the two must be antipodal. Each g_i corresponds to a 15^{th} root of unity, and since 15 is odd, no two points are antipodal. From this we see that P is a simplicial 12-polytope.

If, for our example, we had chosen $j = 5$, which divides 15, the Gale diagram G would be

$$\begin{pmatrix} g_0 \\ \vdots \\ g_{14} \end{pmatrix}$$

where $g_a = \left(\cos(\frac{2\pi a}{3}), \sin(\frac{2\pi a}{3})\right)$. Conv$(G)$ in this case is a regular triangle, each vertex having multiplicity 5. Once again, there is no circuit with less than three such g_i's having nonzero coefficients, all of which are of the same sign.

It turns out (as will be proved in the next section) that the Gale diagram associated to any representation on the $\lfloor \frac{n}{2} \rfloor - 1$ level of the representation lattice is a regular k-gon, each vertex of which has multiplicity (n, j) where $k = \frac{(n,j)}{n}$ and j identifies the sole character V_j not appearing in the representation.

2.2 Duality Theorem

We now prove the main correlation between the various levels of our lattice $\mathcal{L}$, which tells us that for any polytope P corresponding to a representation ρ on level $\lfloor \frac{n}{2} \rfloor - k$ of $\mathcal{L}_n$, its Gale diagram exists as the vertices of a polytope P' associated with a representation $\rho' \in \mathcal{L}_n$ on level k.

Theorem 5. *Let $n \in \mathbb{Z}$, and let P be a multi-cyclic polytope corresponding to a representation ρ on level $\lfloor \frac{n}{2} \rfloor - l$ of the lattice $\mathcal{L}_n$, such that $\rho = V_{k_1} \oplus \cdots \oplus V_{k_m}$ where $K = \{k_1, \ldots, k_m\} \subset \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$. Then the Gale diagram of P consists of the vertices of P', the polytope corresponding to the representation ρ' on level l of $\mathcal{L}_n$ such that $\rho' = V_{k'_1} \oplus \cdots \oplus V_{k'_l}$ where $K' = \{k'_1, \ldots, k'_l\}$ is the complement of K in $\{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$.*

Proof. First, note that for the root of unity $z = e^{\frac{2\pi i}{n}}$, the sum: $1+z+z^2+\cdots+z^{n-1} = \frac{1-z^{n-1}}{1-z} = 0$.

Let the columns of the matrix V be the vertices of P. Then after we linearize V, we have

$$\tilde{V} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \cos(\frac{2\pi k_1 0}{n}) & \cos(\frac{2\pi k_1 1}{n}) & \cdots & \cos(\frac{2\pi k_1 (n-1)}{n}) \\ \sin(\frac{2\pi k_1 0}{n}) & \sin(\frac{2\pi k_1 1}{n}) & \cdots & \sin(\frac{2\pi k_1 (n-1)}{n}) \\ \vdots & \vdots & \ddots & \vdots \\ \cos(\frac{2\pi k_m 0}{n}) & \cos(\frac{2\pi k_m 1}{n}) & \cdots & \cos(\frac{2\pi k_m (n-1)}{n}) \\ \sin(\frac{2\pi k_m 0}{n}) & \sin(\frac{2\pi k_m 1}{n}) & \cdots & \sin(\frac{2\pi k_m (n-1)}{n}) \end{pmatrix}.$$

Let

$$G = \begin{pmatrix} \cos(\frac{2\pi k'_1 0}{n}) & \sin(\frac{2\pi k'_1 0}{n}) & \cdots & \cos(\frac{2\pi k'_l 0}{n}) & \sin(\frac{2\pi k'_l 0}{n}) \\ \cos(\frac{2\pi k'_1 1}{n}) & \sin(\frac{2\pi k'_1 1}{n}) & \cdots & \cos(\frac{2\pi k'_l 1}{n}) & \sin(\frac{2\pi k'_l 1}{n}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \cos(\frac{2\pi k'_1 (n-1)}{n}) & \sin(\frac{2\pi k'_1 (n-1)}{n}) & \cdots & \cos(\frac{2\pi k'_l (n-1)}{n}) & \sin(\frac{2\pi k'_l (n-1)}{n}) \end{pmatrix}.$$

If V is the vertex matrix for some polytope P in $\mathcal{L}_n$, this makes G the vertices of the corresponding 'complement' polytope P'. We have that G is the Gale diagram of P if and only if $\tilde{V}G = 0$.

To this end, the entries in the product matrix $\tilde{V}G$ will be determined in one of four ways.

Case 1 : The product of the i^{th} column of G with the top row of $\tilde{V}$ will be either

$$\sum_{j=0}^{n-1} \cos(\frac{2\pi j k'}{n}) \quad \text{or} \quad \sum_{j=0}^{n-1} \sin(\frac{2\pi j k'}{n}).$$

These are respectively the real and imaginary components of roots of unity, both of which sum to zero from above. So the product is zero.

Case 2 : Entry k, k' will be given by

$$\sum_{j=0}^{n-1} \cos(\frac{2\pi j k}{n}) \cos(\frac{2\pi j k'}{n}).$$

Using basic trigonometric identities, this expands to

$$\frac{1}{2}\sum_{j=0}^{n-1}\left[\cos(\frac{2\pi\cdot j(k-k')}{n})+\cos(\frac{2\pi\cdot j(k+k')}{n})\right]$$

$$=\frac{1}{2}\sum_{j=0}^{n-1}\cos(\frac{2\pi j(k-k')}{n})+\frac{1}{2}\sum_{j=0}^{n-1}\cos(\frac{2\pi j(k+k')}{n}).$$

These two sums correspond to the sums of the real components of the p^{th} and q^{th} roots of unity, respectively, where $p=\frac{n}{(n,k-k')}$ and $q=\frac{n}{(n,k+k')}$. From above, these both equal zero, and hence the entry in the product matrix is zero.

Case 3 : Entry k,k' will be given by

$$\sum_{j=0}^{n-1}\sin(\frac{2\pi kj}{n})\sin(\frac{2\pi k'j}{n}).$$

Again, using trigonometric identities, this expands to

$$\frac{1}{2}\sum_{j=0}^{n-1}\left[\cos(\frac{2\pi\cdot j(k-k')}{n})-\cos(\frac{2\pi\cdot j(k+k')}{n})\right]$$

$$=\frac{1}{2}\sum_{j=0}^{n-1}\cos(\frac{2\pi j(k-k')}{n})-\frac{1}{2}\sum_{j=0}^{n-1}\cos(\frac{2\pi j(k+k')}{n}).$$

As above, both of these sums are zero, and hence the entry in the product matrix is zero.

Case 4 : Entry k,k' will be given by

$$\sum_{j=0}^{n-1}\cos(\frac{2\pi kj}{n})\sin(\frac{2\pi k'j}{n})$$

$$=\frac{1}{2}\sum_{j=0}^{n-1}\left[\sin(\frac{2\pi\cdot j(k+k')}{n})-\sin(\frac{2\pi\cdot j(k-k')}{n})\right]$$

$$=\frac{1}{2}\sum_{j=0}^{n-1}\sin(\frac{2\pi j(k+k')}{n})-\frac{1}{2}\sum_{j=0}^{n-1}\sin(\frac{2\pi j(k-k')}{n}).$$

These are the sums of the imaginary components of the p^{th} and q^{th} roots of unity for p, q as defined above, and hence the sums are both zero.

Therefore, all entries in the product matrix $\tilde{V}G$ are zero, and hence G is the Gale diagram of V.

$\square$

2.3 Applications

We can now apply this duality found within the lattice $\mathcal{L}$ to certain levels in order to classify the polytopes derived from the representations found there.

Going back to the case visited earlier, let P be a polytope corresponding to a representation ρ on level $\lfloor \frac{n}{2} \rfloor - 1$, such that $\rho = V_{k_1} \oplus \cdots \oplus \widehat{V_{k_j}} \oplus \cdots \oplus V_{k_{\lfloor \frac{n}{2} \rfloor}}$ where the k_i's are distinct elements in $\{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$.

From the previous theorem, then, we know that the Gale diagram for P will be either the vertices of a 2-dimensional regular polygon generated by $\rho(a) = (\cos(\frac{2\pi k_j a}{n}), \sin(\frac{2\pi k_j a}{n}))$ for $a \in \mathbb{Z}/n\mathbb{Z}$, or the set $\{-1, 1\}$ if n is even and the missing character is $V_{\frac{n}{2}}$. From this we can state:

Corollary 1. *A polytope P defined as* $\mathrm{conv}(\rho(a))$ *for the representation ρ on the $\lfloor \frac{n}{2} \rfloor - 1$ level of $\mathcal{L}_n$ is simplicial if and only if $\frac{n}{(k_j,n)}$ is odd or equal to 2.*

Proof. Let P be as stated and let G denote the Gale diagram of P. Note first that P is either a d-polytope with $d+2$ vertices (when $n (= d+2)$ is even and $k_j = \frac{n}{2}$), or a d-polytope with $d+3$ vertices (otherwise).

Assume the former and note that $\frac{n}{(k_j,n)} = 2$. Then a non-simplicial facet of P will correspond to a circuit of G with only one non-zero element. But by the previous theorem, G is the set of points $\{(-1)^a \mid a \in \mathbb{Z}/n\mathbb{Z}\}$. Clearly, then, there is no such 1-circuit and hence P is simplicial[3].

Now, assume the latter. In this case, as stated earlier, G consists of the vertices of a regular polygon about the origin. A non-simplicial facet of P will correspond to a circuit of G with one or two non-zero elements (both having the same sign in the case of 2 vertices). Again, there can not be exactly one since no vertex is mapped to the origin (the vertices correspond to roots of unity). Two vertices, g_1 and g_2, will have a dependency exactly when they are antipodal. Clearly, this occurs only when G denotes the vertices of a polygon with an even number of vertices. Viewing the elements of G as roots of unity, the generator, $e^{\frac{2\pi i \cdot k_j}{n}}$ is a $\frac{n}{(n,k_j)}$th root of unity. So antipodal points will exist if and only if $\frac{n}{(n,k_j)}$ is even. Hence, if $\frac{n}{(n,k_j)}$ (or n, obviously) is odd, no two such points will exist and P is simplicial. $\square$

More simply, if n is even and $d = 2^m$ is the highest power of 2 that divides n, then P is simplicial if and only if $d | k_j$ or $k_j = \frac{n}{2}$. If n is odd, P is simplicial. Moreover, the complete face structure of P is evident in that if $k_j = \frac{n}{2}$, then P is one of the well studied cyclic polytopes, and if $k_j \neq \frac{n}{2}$, then the simplicial faces correspond to the complement (in $\mathbb{Z}/n\mathbb{Z}$) of any 3 elements of G that are not contained in a closed semi-circle about the origin. Further, we see easily that the number of distinct combinatorial types on this level of $\mathcal{L}_n$ is equal to the number of (proper) divisors of n.

[3] In fact, such a polytope P is combinatorially equivalent to a cyclic polytope. This follows from Theorem 2.

We saw in the introduction that the multi-cyclic polytope whose vertices are of the form

$$\left(\cos(\frac{2\pi a}{n}), \sin(\frac{2\pi a}{n}), \cos(\frac{4\pi a}{n}), \sin(\frac{4\pi a}{n}), \dots, \cos(\frac{2m\pi a}{n}), \sin(\frac{2m\pi a}{n})\right)$$

for $a \in \mathbb{Z}/n\mathbb{Z}$ is combinatorially equivalent to the cyclic polytope $C_{2m}(n)$, and we referred to them as the trigonometric cyclic polytopes.

So let P be the polytope whose representation is on level $\lfloor \frac{n}{2} \rfloor - 1$ of $\mathcal{L}_n$ and be of the form stated in the previous corollary. Then if n is odd and $k_j = \lfloor \frac{n}{2} \rfloor$, we get a trigonometric cyclic polytope, and since $(k_j, n) = 1$, the Gale diagram consists of the vertices of a regular n-gon. Obviously, if two polytopes have the same Gale diagram, then they are combinatorially equivalent. It follows then that for odd n, all polytopes whose representations are missing only the character V_{k_j} are combinatorially equivalent to $C_{n-3}(n)$ if $(k_j, n) = 1$ since the respective Gale diagram will be the set of vertices of a regular n-gon.

From this is also follows that if n is prime, such P is combinatorially equivalent to $C_{n-3}(n)$ for all k_j since, for prime n, $(n, m) = 1$ for all $m < n$.

2.4 Sums of Roots of Unity

Using this duality we find in the lattice, we have reduced the problem of the face structure of the multi-cyclic polytopes to a problem concerning systems of polynomials in roots of unity over $\mathbb{R}^+$ with a minimal number of non-zero coefficients.

Let $P = \mathrm{conv}(V)$ be a multi-cyclic polytope whose representation

$$\rho = V_{k_1} \oplus \cdots \oplus V_{k_m}$$

is on level $\lfloor \frac{n}{2} \rfloor - i$ of $\mathcal{L}_n$. By Theorem 5, then, the Gale diagram G of P consists of the vertices of P' corresponding to the representation $\rho' = V_{k'_1} \oplus \cdots \oplus V_{k'_l}$ where, as above, the set of all $V_{k'_j}$ is the set of characters missing from the representation ρ.

Let $S = \{a_1, \dots, a_s\} \subset \{0, 1, \dots, n-1\}$ and $S' = \{a'_1, \dots, a'_{s'}\}$ be the complement of S in $\{0, \dots, n-1\}$. Let also $g_{a_i} = \rho'(a_i)$ and $v_{a_i} = \rho(a_i)$ for $a_i \in \mathbb{Z}/n\mathbb{Z}$. Then for a subset $g_{a_1}, \dots, g_{a_s}$ of G, if there exist scalars $\lambda_1, \dots, \lambda_s$ in $\mathbb{R}^+$ such that $\sum_{j=1}^{s} \lambda_j g_{a_j} = 0$ is minimal, then the set $\{v_{a'_1}, \dots, v_{a'_{s'}}\} \subset V$ forms a facet of P. If P is a d-polytope with n vertices, then for such sets, $s \leq n - d$.

Expanding this slightly, let $\{g_{a_1}, \dots, g_{a_s}\}$ be such a set. Then we want $\lambda_1, \dots, \lambda_s$ so that for $\theta = \frac{2\pi}{n}$,

$$\sum_{j=1}^{s} \lambda_j (\cos(\theta a_j k'_1), \sin(\theta a_j k'_1), \dots, \cos(\theta a_j k'_l), \sin(\theta a_j k'_l)) = 0 \qquad (2.1)$$

where k'_i refers to the representation $V_{k'_i}$ appearing in ρ'. If n is even and $V_{\frac{n}{2}}$ appears in the sum defining ρ', then one entry (say, the first one) of (2.1) will be

$(-1)^{a_j}$. Grouping this sum component-wise and replacing $(\cos(\theta a_j k'_j), \sin(\theta a_j k'_j))$ with $e^{\frac{2\pi}{n} a_j k'_j}$, we then have, for $\omega = e^{\frac{2\pi i}{n}}$:

$$\begin{aligned} (\lambda_1(-1)^{a_1} + \lambda_1(-1)^{a_2} + \cdots + \lambda_1(-1)^{a_s} &= 0) \\ \lambda_1 \omega^{k'_1 a_1} + \lambda_2 \omega^{k'_1 a_2} + \cdots + \lambda_s \omega^{k'_1 a_s} &= 0 \\ &\vdots \\ \lambda_1 \omega^{k'_l a_1} + \lambda_2 \omega^{k'_l a_2} + \cdots + \lambda_s \omega^{k'_l a_s} &= 0. \end{aligned}$$

So, we want $(\lambda_1, \ldots, \lambda_s) \in (\mathbb{R}^+)^s$ such that

$$\begin{pmatrix} (-1)^{a_1} & (-1)^{a_2} & \cdots & (-1)^{a_s} \\ \omega^{k'_1 a_1} & \omega^{k'_1 a_2} & \cdots & \omega^{k'_1 a_s} \\ \vdots & \vdots & \ddots & \vdots \\ \omega^{k'_l a_1} & \omega^{k'_l a_2} & \cdots & \omega^{k'_l a_s} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_s \end{pmatrix} = 0.$$

Note that the first row of this matrix exists only when applicable for the obvious reasons.

Phrasing the problem in this way, let us apply the relation defined in Theorem 5 to level $\lfloor \frac{n}{2} \rfloor - 2$ and level 2 of $\mathcal{L}_n$. The representations on level 2 correspond to either 3-dimensional polyhedra or the 4-dimensional bi-cyclic polytopes studied by Smilansky, both of which are understood. So let P be the polytope corresponding to the representation ρ on level $\lfloor \frac{n}{2} \rfloor - 2$ of $\mathcal{L}_n$, which will be the complete representation minus two characters χ_{k_1} and χ_{k_2}. Then the Gale diagram of P will consist of the vertices of the polytope P' for the representation

$$\rho'(a) = ((-1)^a, \cos(\theta k_1 a), \sin(\theta k_1 a)) \tag{2.2}$$

or

$$\rho'(a) = (\cos(\theta k_1 a), \sin(\theta k_1 a), \cos(\theta k_2 a), \sin(\theta k_2 a)) \tag{2.3}$$

for $k_1, k_2 \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$ and $\theta = \frac{2\pi}{n}$.

Theorem 6. *For a multicyclic polytope P whose Gale diagram is given by* (2.2)*, P is non-simplicial if and only if $\frac{n}{(n,k_1)} \equiv 2 \bmod 4$, in which case all non-simplicial facets have $n-2$ vertices.*

Proof. P is an $(n-4)$-polytope, and so is non-simplicial if and only if there exist circuits of G, the Gale diagram of P, with less than 4 elements. Obviously, there are no 1-circuits for G.

For 2-circuits, we want $a_1, a_2 \in \mathbb{Z}/n\mathbb{Z}$ such that

$$\begin{pmatrix} (-1)^{a_1} & (-1)^{a_2} \\ \omega^{k_1 a_1} & \omega^{k_1 a_2} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0$$

for $\lambda_1, \lambda_2 \in \mathbb{R}^+$. We may assume that $a_1 = 0$, giving

$$\begin{pmatrix} 1 & (-1)^{a_2} \\ 1 & \omega^{k_1 a_2} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0$$

For λ_1, λ_2 to be positive, $(-1)^{a_2}$ must be -1, so $a_2 \equiv 1 \bmod 2$. Then we may assume $\lambda_1 = \lambda_2 = 1$, implying that $\omega^{k_1 a_2} = -1$, and so clearly $\frac{n}{(n,k_1)}$ is even and $k_1 a_2 \equiv \frac{n}{2} \bmod n$. Then $2k_1 a_2 \equiv 0 \bmod n$ and $2a_2 \equiv 0 \bmod \frac{n}{d}$ where $d = (n, k_1)$. Since we know that $\frac{n}{d}$ is even, it follows that $a_2 \equiv 0 \bmod \frac{n}{2 \cdot d}$. Since a_2 is odd, we know from this that $\frac{n}{2 \cdot d}$ is odd, and hence $\frac{n}{(n,k_1)} \equiv 2 \bmod 4$. The complement of such a_1, a_2 for these circuits gives the $n-2$ vertices of the non-simplicial facets.

To prove that these are the only non-simplicial facets we must show that for ρ', there are no 3-circuits.

To this end, assume otherwise, that there exist $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}^+$ such that $\lambda_1 \rho(a') + \lambda_2 \rho(a'') + \lambda_3 \rho(a''') = 0$. Since $\rho'(a) = ((-1)^a, \cos(\frac{2\pi a k_1}{n}), \sin(\frac{2\pi a k_1}{n}))$, then for all λ_i to be positive, exactly one element, say $\rho'(a')$, has distinct first entry from the other two. Then subtracting a' from each such a gives us elements $\rho'(a_0) = \rho'(0) = (1,1,0)$ and $\rho'(a_1), \rho'(a_2)$ each with first entry equal to -1. By dividing the linear dependency through by λ_1, we must now solve

$$(1,1,0) + c_1(-1, y_1, z_1) + c_2(-1, y_2, z_2) = 0.$$

For $c_1, c_2 > 0$, we have immediately that

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 y_1 + c_2 y_2 &= -1 \\ c_1 z_1 + c_2 z_2 &= 0 \end{aligned}$$

Which gives us that $c_2 = 1 - c_1$ and that

$$y_1 = \frac{c_1 y_2 - y_2 - 1}{c_1} \quad \text{and} \quad z_1 = \frac{c_1 z_2 - z_2}{c_1} \tag{2.4}$$

Now, note that $y_1^2 + z_1^2 = 1$ and $y_2^2 + z_2^2 = 1$. By (2.4), the first becomes

$$\left(\frac{c_1 y_2 - y_2 - 1}{c_1}\right)^2 + \left(\frac{c_1 z_2 - z_2}{c_1}\right)^2 = 1,$$

which is

$$c_1^2(y_2^2 + z_2^2) - 2c_1(y_2^2 + z_2^2 + y_2) + (y_2^2 + z_2^2 + 2y_2 + 1) = c_1^2.$$

Since $y_2^2 + z_2^2 = 1$, we have

$$c_1^2 - 2c_1(y_2 + 1) + 2(y_2 + 1) = c_1^2, \text{ so } c_1 = \frac{-2(y_2+1)}{-2(y_2+1)} = 1$$

and hence $c_2 = 0$. So $\rho'(a_1)$ is a scalar multiple of $\rho'(a_0)$; but more importantly we see that $\{\rho'(a_0), \rho'(a_1), \rho'(a_2)\}$ reduces to a 2-cycle. $\square$

Now, let P a multi-cyclic polytope such that its Gale diagram is given by a representation of the form in (2.3).

So given

$$G = \begin{pmatrix} \omega^{0\cdot k_1} & \omega^{1\cdot k_1} & \dots & \omega^{(n-1)\cdot k_1} \\ \omega^{0\cdot k_2} & \omega^{1\cdot k_2} & \dots & \omega^{(n-1)\cdot k_2} \end{pmatrix}$$

for $\omega = e^{\frac{2\pi i}{n}}$, our goal then is to find $2 \times s$ submatrices whose kernel contains an element in $(\mathbb{R}^+)^s$. We can confine our analysis to the set $(s = \{2, 3, 4, 5\})$; if s is greater, then the complement of the vertex set can not be a full-dimensional facet of P. We shall work through the possibilities for each such value of s.

Theorem 7. *For a polytope P whose Gale diagram is given by* (2.3), *(so that P is an $(n-5)$-polytope): If n is odd, then there are no facets with exactly $n-2$ vertices. Facets with exactly $n-3$ vertices exist if and only if $d = (k_2 - k_1, n) \geq 3$. In this case the three vertices not contained in such a facet correspond to distinct elements a_1, a_2, a_3 all contained in a left coset of the unique subgroup of order d of $\mathbb{Z}/n\mathbb{Z}$. If n is even, facets with exactly $n-2$ vertices correspond to the complement of some distinct a_1 and a_2 contained in some left coset of the subgroup of order $2d$ for $d = (k_2 - k_1, n)$ if $d \geq 2$. Facets with exactly $n-3$ vertices correspond to cosets of the subgroup of order d where $d = (k_2 \pm k_1, n) \geq 3$.*

Proof. For such 2-circuits, let

$$M = \begin{pmatrix} \omega^{k_1 a_1} & \omega^{k_1 a_2} \\ \omega^{k_2 a_1} & \omega^{k_2 a_2} \end{pmatrix}.$$

M will have nontrivial kernel if and only if $\operatorname{rank}(M) = 1$, meaning that the two rows are dependent, and hence the desired kernel is the set $\{(x, y) \mid x \cdot \omega^{k_1 a_1} + y \cdot \omega^{k_1 a_2} = 0 \text{ for } x, y \in \mathbb{R}^+\}$. Since $\omega^{k_1 a_1}, \omega^{k_1 a_2}$ are roots of unity, obviously they must be antipodal. From this we see that $\frac{n}{(k_1,n)}$ (and hence n) is even. Since the two points must be antipodal, we know that $k_1 a_1 \equiv k_1 a_2 + \frac{n}{2} \bmod n$, and then $2k_1 a_1 - 2k_1 a_2 \equiv 0 \bmod n$. Similarly for the second row, we get $2k_2 a_1 - 2k_2 a_2 \equiv 0 \bmod n$. Combining these yields that $a_1 \equiv a_2 \bmod \frac{n}{2d}$ for $d = (k_2 - k_1, n)$. Hence, such elements a_1 and a_2 must be subsets of the left cosets xH of H, the subgroup of order d.

For 3-circuits of P', let now

$$M = \begin{pmatrix} \omega^{k_1 a_1} & \omega^{k_1 a_2} & \omega^{k_1 a_3} \\ \omega^{k_2 a_1} & \omega^{k_2 a_2} & \omega^{k_2 a_3} \end{pmatrix}.$$

First let us assume that $\operatorname{rank}(M) = 2$. By minors then,

$$\ker(M) = \begin{pmatrix} \omega^{k_1 a_2 + k_2 a_3} - \omega^{k_1 a_3 + k_2 a_2} \\ \omega^{k_1 a_1 + k_2 a_3} - \omega^{k_1 a_3 + k_2 a_1} \\ \omega^{k_1 a_1 + k_2 a_2} - \omega^{k_1 a_2 + k_2 a_1} \end{pmatrix}.$$

For $z = re^{i\varphi} \in \mathbb{C}$, $r \neq 0$, we want elements in the kernel of M such that

$$e^{i\varphi} \begin{pmatrix} \omega^{k_1 a_2 + k_2 a_3} - \omega^{k_1 a_3 + k_2 a_2} \\ \omega^{k_1 a_1 + k_2 a_3} - \omega^{k_1 a_3 + k_2 a_1} \\ \omega^{k_1 a_1 + k_2 a_2} - \omega^{k_1 a_2 + k_2 a_1} \end{pmatrix} \in \mathbb{R}^3.$$

For these entries to be real-valued, we need:

$$\sin((k_1a_2 + k_2a_3)\theta + \varphi) = \sin((k_1a_3 + k_2a_2)\theta + \varphi)$$
$$\sin((k_1a_1 + k_2a_3)\theta + \varphi) = \sin((k_1a_3 + k_2a_1)\theta + \varphi)$$
$$\sin((k_1a_1 + k_2a_2)\theta + \varphi) = \sin((k_1a_2 + k_2a_1)\theta + \varphi)$$

for $\theta = \frac{2\pi}{n}$, $0 \leq \varphi \leq 2\pi$.

So then we have congruencies of the form:

$$k_1a_2 + k_2a_3 + \varphi \equiv k_1a_3 + k_2a_2 + \varphi \bmod n$$
$$k_1a_1 + k_2a_3 + \varphi \equiv k_1a_3 + k_2a_1 + \varphi \bmod n$$
$$k_1a_1 + k_2a_2 + \varphi \equiv k_1a_2 + k_2a_1 + \varphi \bmod n$$

or

$$k_1a_2 + k_2a_3 + \varphi \equiv \tfrac{n}{2} - (k_1a_3 + k_2a_2 + \varphi) \bmod n$$
$$k_1a_1 + k_2a_3 + \varphi \equiv \tfrac{n}{2} - (k_1a_3 + k_2a_1 + \varphi) \bmod n$$
$$k_1a_1 + k_2a_2 + \varphi \equiv \tfrac{n}{2} - (k_1a_2 + k_2a_1 + \varphi) \bmod n.$$

The first set of congruencies easily yields that $a_1 \equiv a_2 \equiv a_3 \bmod \frac{n}{d}$, where $d = (k_2 - k_1, n)$. Further, since a_1, a_2, a_3 are distinct in $\mathbb{Z}/n\mathbb{Z}$, $d \geq 3$. Let H be the subgroup of order $\frac{n}{d}$ in $\mathbb{Z}/n\mathbb{Z}$. Then a_1, a_2, and a_3 are again elements of the claimed cosets.

For the second set of congruencies to apply, n must obviously be even. Subtracting one from another in turn gives $k_1a_2 - k_1a_1 \equiv -k_2a_2 + k_2a_1 \bmod n$, etc, which then give that $a_1 \equiv a_2 \equiv a_3 \bmod \frac{n}{d}$ for $d = (k_2 + k_1, n)$.

Now, let $\operatorname{rank}(M) = 1$. For the rows of M to be dependent, we can claim that: $(\omega^{k_2a_1}, \omega^{k_2a_2}, \omega^{k_2a_3}) = \pm\zeta(\omega^{k_1a_1}, \omega^{k_1a_2}, \omega^{k_1a_3})$ where ζ is an n^{th} root or unity. Then $\omega^{k_2a_i} = \pm(\omega^{k_1a_i+x})$ for some $x \in \{0, \ldots, n-1\}$. Then $k_2a_i = \pm(k_1a_i + x) \bmod n$, and solving shows that $a_1 \equiv a_2 \equiv a_3 \equiv \pm x' \bmod \frac{n}{d}$ for $x' = \frac{x}{(x,d)}$ and $d = (k_2 \pm k_1, n)$. Once again, d must be at least 3 for the a_i's to be distinct mod n. All a_i's are again in the corresponding coset, and the theorem is completed. $\square$

Only certain subsets of the cosets of the above subgroups correspond to circuits; note that we have only made restrictions on the elements of the circuits based on the coefficients of the linear dependencies being real-valued and not necessarily positive. If $d \geq 5$, there are subsets $\{a_1, a_2, a_3\}$ that when multiplied by say, k_1, all lie on a closed semi-circle, and hence some coefficient of the linear dependency will be negative.

Illustrating with a concrete example, let $n = 15$, $k_1 = 2$, and $k_2 = 5$. Then non-simplicial facets will correspond to subsets of the cosets of the subgroup of order 3, the elements of which are $\{0, 5, 10\}$. These subsets are exactly $(a_1, a_2, a_3) = \{(0, 5, 10), (1, 6, 11), \ldots, (4, 9, 14)\}$. Since $d = 3$, all of these 3-tuples are circuits of

G, showing that P is a 10-polytope with 5 nonsimplicial faces whose vertices are the complements of the 3-tuples in the solution set above.

The exact classification of the 4-circuits of $\rho'(a)$ are not solved, though the problem can be restated (and reduced, hopefully). Let

$$M = \begin{pmatrix} \omega^{k_1a_1} & \omega^{k_1a_2} & \omega^{k_1a_3} & \omega^{k_1a_4} \\ \omega^{k_2a_1} & \omega^{k_2a_2} & \omega^{k_2a_3} & \omega^{k_2a_4} \end{pmatrix}.$$

First, I claim that for $(a_1, \ldots, a_4)$ to be a minimal circuit, $\operatorname{rank}(M) = 2$. To see this, note that if the two rows of M are a dependent set, then the second row can be written as $\pm(\omega^{k_1a_1}, \omega^{k_1a_2}, \omega^{k_1a_3}, \omega^{k_1a_4})$ or $\zeta(\omega^{k_1a_1}, \omega^{k_1a_2}, \omega^{k_1a_3}, \omega^{k_1a_4})$ where ζ is a n^{th} root of unity, both of which imply that the second row has the same "shape" as the first (the entries of the second row have the same position relative to each other as those on the first row). Then simply note that for any four points on a circle that are cyclic, some proper subset of them must also be cyclic and hence the circuit is not minimal.

So assume $\operatorname{rank}(M) = 2$. Then

$$\ker(M) = \operatorname{span}\begin{pmatrix} \omega^{k_1a_2+k_2a_4} - \omega^{k_1a_4+k_2a_2} & \omega^{k_1a_2+k_2a_3} - \omega^{k_1a_3+k_2a_2} \\ \omega^{k_1a_1+k_2a_4} - \omega^{k_1a_4+k_2a_1} & \omega^{k_1a_1+k_2a_3} - \omega^{k_1a_3+k_2a_1} \\ 0 & \omega^{k_1a_1+k_2a_2} - \omega^{k_1a_2+k_2a_1} \\ \omega^{k_1a_1+k_2a_2} - \omega^{k_1a_2+k_2a_1} & 0 \end{pmatrix}$$

To find a 4-circuit, we are looking for some linear combination of the two columns of the kernel matrix to be in $(\mathbb{R}^+)^4$. So we will start with the entries on the bottom two rows, which are equal. For these to be real, (as above we will assume $a_1 = 0$) we once again want solutions to one of the following congruencies: $k_2a_2 \equiv k_1a_2 \bmod n$ or $k_2a_2 \equiv \frac{n}{2} - k_1a_2 \bmod n$

The reduction of the problem to vanishing sums of roots of unity has some references in the literature. Though it is more concerned with the fields in which the coefficients of roots of unity lie, there is some independent interest in the problem itself; see [2] and its bibliography.

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