Extending the Critical Group to Oriented Matroids and Simplicial Complexes

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Abstract

In this thesis, we extend the definition of the critical group of a graph to the class of oriented matroids (OMs). We also discuss abstract simplicial compelxes (ASC's), and their critical groups. Matroid complexes are ASC's with facets taken from the bases of a matroid. Stanley's Conjecture states that the h-vector of a matroid complex is a pure O-sequence.

Oriented matroids are used to study among other things, point configurations in $\mathbb{R}^n$. We use such OMs to examine some basic properties of the critical group defined on OMs. We classify the critical group for the class of uniform matroids and show it is independent of orientation. The canonical free join, $X \bowtie Y$, of two point configurations X and Y has the property that $\mathrm{Crit}(X \bowtie Y) = \mathrm{Crit}(X) \times \mathrm{Crit}(Y)$, where $\mathrm{Crit}(X)$ is the critical group of the OM generated by X.

We also have a conjecture about translating between point configurations taken from hyperplane arrangements, and graphs, via an intermediary hypergraph.

Introduction

In this thesis, we define and investigate extensions of the critical group and sandpile group from the category of graphs to the category of Abstract Simplicial Complexes (ASC's) and the category of Oriented Matroids (OMs). The first chapter is the graph theoretical version, the second chapter is ASC's, the third is OMs, the fourth investigates relations between OMs and ASC's, involving the Tutte polynomial and Stanley's conjecture on the h-vector of a matroid complex, and the fifth involves using hypergraphs to translate between representations of OMs as graphs and point configurations.

I use **boldface** when making a definition. I sometimes use *italics* for terms which have formal meanings when I am not defining them (I also use them for emphasis). I use "quotations" for informal notions.

Chapter 1

Graphs

A **simple, undirected graph**, or simply a **graph**, G, is a set of **vertices**, $V = V(G)$, and a set of **edges**, or pairs of vertices, $E = E(G)$.

$$G = \{V, E\}.$$

The adjective "undirected" is in contrast to **directed graphs** (or **digraphs**), in which the edges are *ordered pairs*, $e = \{e^-, e^+\}$, with a **head,** e^-, and a **tail**, e^+. "Simple" is in contrast to **multigraphs**, in which the edge set, $E(G)$ is a multiset, and **weighted graphs,** in which every edge is assigned a weight in the real numbers, integers, or natural numbers. A weighting is encoded in the **weight function** of the graph, $\mathrm{wt}_G : E(G) \longrightarrow \mathbb{R}$. We can easily define a weighted graph from any multigraph by taking the underlying graph and assigning weights to the edges according to their multiplicity.

Another subtle distinction can be made (more or less rigorously) between directed multigraphs and "labeled directed multigraphs." Formally, a labeled directed multigraph is a directed multigraph together with a function $\mathrm{Label} : E(G) \cup V(G) \longrightarrow \mathfrak{A}$, for some alphabet of characters, $\mathfrak{A}$. Of course, this definition extends to graphs, and this allows us to identify weighted graphs and digraphs with special types of labeling schemes. The weighted graph Label function just assigns the weights to the edges (as characters, to be precise). For a digraph, we first chose a total ordering of V, and define

$$\mathrm{Label}(\{e_v, e_u\}) = \begin{Bmatrix} +1 \text{ if } v \prec u \\ -1 \text{ if } u \prec v \end{Bmatrix}.$$

The Sandpile Group can be defined for directed multigraphs, but for the majority of this paper, we will only be concerned with simple, undirected graphs.

For a digraph, a multigraph, a labeled graph, or a weighted graph, G, we get the **underlying graph**, $\underline{G}$ by ignoring the direction, label, weight, or number of edges connecting two vertices. This is the simple undirected graph we get by ignoring the extra information attached to edges, so the edges of $\underline{G}$ are the pairs $\{u, v\}$ such that $\{u, v\}$ is an edge of G, ignoring the order for digraphs. We say that a digraph, D, is an **orientation** of G, if G is the underlying graph of D, and every edge of G corresponds to only one edge of D. In this case, we also say that D is an **oriented graph**.

If our vertex set is endowed with a total order, $\prec$, then there is a unique orientation of G, which we will call **the orientation** of G given by taking the smaller vertex as the first element of every edge. The corresponding Label function is just the identity.

1.1 Cuts and Cycles

A **signed cut**, $\{A, A^c\}$ is a partition of the vertices in a graph into two (non-empty) sets; we identify the oriented cuts with the subsets of the vertices, $A \subset V$. Each oriented cut defines a corresponding **cutset**, $\underline{c}(A)$, this is the set of edges going between the two vertex sets of the partition. In the case of an oriented graph, we define the corresponding **signed cutset**, $c(A) := \{\underline{c}(A), f\}$, a signed set, with:

$$f(e) = \left\{ \begin{array}{l} +1 \text{ if } e^+ \in A \\ -1 \text{ if } e^- \in A^c \end{array} \right\}.$$

The set of all signed cutsets of graph, G, we denote $\mathcal{C}^*(G)$. Dual to the signed cuts are the signed cycles of the graph, the set of which is denoted $\mathcal{C}(G)$. A **cycle** is a sequence of vertices, $\{v_0, v_1, ..., v_n\}$, such that $v_i = v_j$ iff $i = j \pmod{n}$, and such that v_i, v_{i+1} are connected for all i.

1.2 The Sandpile Group

This thesis began as an investigation into the sandpile group of a graph. I ended up focusing on the critical group, however, defined in the next section. As a reference for sandpiles, see [25].

1.3 The Critical Group

The critical group of a graph is a graph invariant that (among other things) counts the number of spanning trees of the graph (this is the order of the critical group). In the case of undirected graphs, the sandpile group doesn't depend on choice of sink, and it is isomorphic to the critical group. We define the critical group as

$$\mathrm{Crit}(G) = \mathbb{Z}E/(\mathfrak{C} + \mathfrak{C}^*),$$

where $\mathfrak{C} = \mathrm{Span}_{\mathbb{Z}}(\mathcal{C})$, $\mathfrak{C}^* = \mathrm{Span}_{\mathbb{Z}}(\mathcal{C}^*)$, are the **cycle space** and **cut space**, of the graph, respectively.

Chapter 2

Abstract Simplicial Complexes

A graph is a special case of a more general object: the simplicial complex. A **simplicial complex** is a topological space constructed by gluing *simplices* together along *faces*. This amounts to imposing the following restrictions on a family of simplices, $\mathcal{K}$:

* every face of every member of $\mathcal{K}$ is in $\mathcal{K}$.

* for all $\sigma_1, \sigma_2 \in \mathcal{K}$, $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

We are more concerned here with **abstract simplicial complexes (ASC's)**. We think of ASC's as "unrealized" simplicial complexes; lacking an embedding and hence geometric features, ASC's are purely combinatorial objects. To form an ASC, Δ, we start with a finite set of **vertices**, $V = V(\Delta) = \{v_1, .., v_n\}$. The ASC consists of a set of subsets of V closed under the taking of subsets, that is,

$$X \subseteq \Delta \implies Y \subseteq \Delta, \forall Y \subset X.$$

The members of Δ are called **faces**. A maximal face is called a **facet**. We also order the vertices via $v_i \prec v_j$ iff $i < j$.

The **dimension** of a face, $X \in \Delta$, is $|X| - 1$. The **dimension** of Δ is the dimension of its largest face(s). In the case that all facets have the same size, we say that Δ is **pure**.

Every ASC can be geometrically realized as a simplicial complex. This involves identifying every face, $X \in \Delta$ of dimension k, with a (topological) simplex of dimension k, so that the incidences and containments of Δ are preserved in the resulting simplicial complex, $\mathcal{K}$.

A (simple, undirected) graph is naturally an ASC of dimension ≤ 1. For a graph G, the set $\Delta = V(G) \cup E(G) \cup \emptyset$ is the associated ASC. For any ASC, Δ, its **i-dimensional skeleton,** denoted $\Delta_{(i)}$, is the ASC we get by restricting to faces of dimension $\leq i$. The **skeleton** or **underlying graph** of an ASC is just its 1-dimensional skeleton (a graph). We use Δ_i to refer to the set of i-dimensional faces of Δ. Notice that, in general,

$$\Delta_i \neq \Delta_{(i)}.$$

2.1 Boundaries and Homology

Abstract Simplicial Complexes are naturally equipped with a **boundary map**, $\partial = \sum_{i=0}^{\infty} \partial_i$. We typically restrict the domain of the ∂_i's to be $\mathbb{Z}$-linear combinations of the i-dimensional faces of Δ, as they are identically 0 elsewhere. Each ∂_i is a linear function, determined by its action on the faces, $f = \{f_1, .., f_i\}$, $f_j \prec f_{j+1}$, $\forall j$:

$$\partial_i(\{f_1, ..., f_i\}) = \sum_{l=1}^{i} (-1)^{l-1}\{f_1, ..., f_{l-1}, f_{l+1}, ..., f_i\}.$$

We can represent ∂_i as a matrix with columns representing i-dimensional faces, and rows corresponding to faces of dimension $i-1$. This makes it easy to define the **coboundary map of dimension i**, denoted ∂_i^* to be the the linear map given by the transpose of the matrix for the boundary map.

The sets of i-dimensional faces of Δ, together with the boundary maps are an example of a chain complex. A **chain complex** is a sequence of abelian groups or modules, $C_0, C_1, ...$ and a sequence of homomorphic boundary operators, $\partial_0, \partial_1, ...$ with each $\partial_n \colon C_n \to C_{n-1}$. We represent chain complexes with diagrams:

$$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

substituting $C_i = \Delta_i$, gives the following diagram:

$$\cdots 0 \longrightarrow 0 \longrightarrow \mathbb{Z}^{\Delta_d} \xrightarrow{\partial_d} \mathbb{Z}^{\Delta_{d-1}} \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_2} \mathbb{Z}^{\Delta_1} \xrightarrow{\partial_1} \mathbb{Z}^{\Delta_0} \xrightarrow{\partial_0} 0$$

For any chain complex (here an ASC, Δ), we can define the **homology groups** of Δ.

$$H_n(\Delta) = \ker(\partial_n)/\operatorname{im}(\partial_{n+1}).$$

A slight modification gives the **reduced homology groups**, which we will use for the remainder of this thesis. Here, we augment our chain complex, by inserting a copy of $\mathbb{Z}$:

$$\cdots 0 \longrightarrow 0 \longrightarrow \mathbb{Z}^{\Delta_d} \xrightarrow{\partial_d} \mathbb{Z}^{\Delta_{d-1}} \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_2} \mathbb{Z}^{\Delta_1} \xrightarrow{\partial_1} \mathbb{Z}^{\Delta_0} \xrightarrow{\Sigma} \mathbb{Z} \longrightarrow 0$$

and exchanging the sum mapping,

$$\begin{aligned} \Sigma : \Delta_0 &\longrightarrow \mathbb{Z} \\ v_i &\mapsto 1. \end{aligned}$$

for $\partial_0 := 0$. This again forms a chain complex, and the homology groups of this chain complex are the *reduced* homology groups of Δ, denoted $\tilde{H}_i(\Delta)$.

The i-th Betti number is $\beta_i(\Delta) = \operatorname{rank}(\tilde{H}_i(\Delta))$. The Betti numbers provide a measurement of topological connectedness properties of a space. The 0-th Betti number is exactly the number of connected components of a space, while i-th Betti numbers measure the number of $i+1$ dimensional "holes" in the space.

2.2 The f-vector and the h-vector

We also define the f-vector and h-vector of an ASC. The **f-vector** is

$$f(\Delta) := (f_{-1}(\Delta), f_0(\Delta), ..., f_{d-1}(\Delta)),$$

where $f_i(\Delta) := |\Delta_i|$. The **h-vector,**

$$h(\Delta) := (h_0(\Delta), ..., h_d(\Delta)),$$

is a recombination of the f-vector, given by:

$$h_k(\Delta) = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}.$$

We also have an explicit formula for the f-vector given the h-vector; these two uniquely determine one another. In fact, the h-vector and f-vector correspond to different evaluations of the same generating function,

$$f_\Delta(x) = \sum_{i=0}^{d} f_{i-1}(\Delta) x^{d-i}.$$

$$h_\Delta(x) = f_\Delta(x-1) = \sum_{i=0}^{d} h_i(\Delta) x^{d-i}.$$

The $f_\Delta(X)$ is called the **face enumerator**, while $h_\Delta(x)$ is the **shelling polynomial**.

2.3 The Critical Group

Duval, Klivens, and Martin recently extended the definition of the critical group and the sandpile group to simplicial complexes [2],[3],[4], in the next two sections, we summarize this work, defining the d-dimensional Critical Group and Sandpile Group of an ASC and show that these extend the graph theory definitions. First, we define the d-dimensional Laplacian, $L_d = \partial_d \partial_d^*$. The d-dimensional Critical Group of an ASC, Δ, is then:

$$\mathrm{Crit}_d(\Delta) = \ker(\partial_d) / \operatorname{im}(L_d).$$

This replicates the definition of the critical group in the graphic case.

2.4 The Sandpile Group

An ASC is **acyclic in positive codimension (APC)** iff

$$\beta_i(\Delta) = 0, \ \forall\, i < d.$$

For an APC ASC, we can also define the **d-dimensional Sandpile Group of $\boldsymbol{\Delta}$**, with respect to a given $(d-1)$-dimensional **spanning complex**, Γ. The choice of sink in the graphical case is generalized by the choice of spanning complex. We denote this group by

$$\mathcal{S}_d(\Delta, \Gamma).$$

For a d-dimensional ASC, Δ, a **spanning complex** of Δ is a connected, acyclic subcomplex of dimension $d-1$ which "spans" Δ. Specifically, Γ is a spanning complex iff $\Gamma_{(d-1)} = \Delta_{(d-1)}$, and Γ satisfies the following conditions:

* $\tilde{H}_d(\Gamma, \mathbb{Z}) = 0$
* $|\tilde{H}_{d-1}(\Gamma, \mathbb{Z})| < \infty$
* $f_d(\Gamma) = f_d(\Delta) - \beta_d(\Delta) + \beta_{d-1}(\Delta)$,

any 2 of which imply the 3rd (so long as $\Gamma_{(d-1)} = \Delta_{(d-1)}$).

Just as the sandpile group of a graph is isomorphic to integer combinations of the non-sink vertex modulo the Laplacian, we now take integer combinations of the edges not in the "sink", Γ, modulo the (now order d) Laplacian:

$$\mathcal{S}_d(\Delta, \Gamma) \approx \mathbb{Z}\Gamma^c / \tilde{L}_d.$$

Of course, technically we need to restrict the Laplacian's domain to $\mathbb{Z}\Gamma^c$, creating the **reduced Laplacian**, $\tilde{L}_d$.

2.4.1 The Sandpile Group and the Critical Group are Isomorphic

We can easily demonstrate the equivalence of these groups for the $d = 2$ case (recall $d = 1$ is the graphic case). Here, the "sink" spanning complex is just a spanning tree for the skeleton of the complex. We know that for any spanning tree, Γ, of a graph, G, adding any edge, e, of G not already included in Γ will create a unique cycle, c_e, containing e. Also, for $d = 2$, $\ker(\partial_{d-1}) = \ker(\partial_1)$ is exactly the cycle space of the underlying graph, which is generated by the c_e's, for fixed Γ, so we get an isomorphism:

$$\begin{aligned} \mathbb{Z}\Gamma^c &\longrightarrow \ker(\partial_1) \\ e &\mapsto c_e. \end{aligned}$$

This allows us to fill in the following commutative diagram, guaranteeing us the desired isomorphism:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}E/\ker(L_2) & \xrightarrow{L_2} & \ker(\partial_1) & \longrightarrow & \mathrm{Crit}(\Delta) & \longrightarrow & 0 \\ & & \uparrow & & {\scriptstyle e\mapsto c_e}\uparrow\approx & & \uparrow\approx & & \\ 0 & \longrightarrow & \mathbb{Z}^{\Gamma^c} & \xrightarrow{\tilde{L}_2} & \mathbb{Z}^{\Gamma^c} & \longrightarrow & \mathcal{S}(\Delta, \Gamma) & \longrightarrow & 0 \end{array}$$

This diagram is really just expressing the fact that it doesn't matter whether we mod out by the Laplacian or the reduced Laplacian.

2.4.2 Firing Rules and Group Representatives

The choice of a "sink" spanning complex gave us the sandpile group of a simplicial complex, $\mathcal{S}_d(\Delta, \Gamma)$, but while the firing rules in the graph case conserve the total amount of "sand", the higher dimensional reduced Laplacians used with ASC's do not necessarily have this property. Furthermore, while in the graphic case, "firing" a vertex always resulted in a less sand on that vertex and only that vertex. However, in the ASC case, "firing" a simplex can decrease the sand on other simplices, so it becomes more difficult to create a sensible notion of stabilization, which we require in order to find canonical group representatives for the equivalence classes created by the action of the reduced Laplacian. However, since the configurations always form a lattice ideal, I, we can choose a Gröbner Basis, $\mathcal{G}$ for I and reduce with respect to $\mathcal{G}$. So any choice of a Gröbner Basis gives a set of unique representatives for the sandpile group. But without the intuitiveness of the graphic stable configurations, these representatives are kind of arbitrary and weren't of any further immediate interest.

2.5 Group Dependence on Choice of Triangulation

In [2], the authors extend the sandpile group to cubical complexes. I spent some time looking for "(abstract) polygonal complexes" in the literature, which would be polyhedra glued together nicely, but I couldn't find them easily and didn't spend too long on this topic. However, one interesting conjecture, to which I was able to produce a counterexample is whether two different triangulations of the same "polygonal complex" would have the same critical group. The "hollow cube" provides a counterexample. The hollow cube is 6 "squares" glued together with 12 lines at 8 points, as below:

$$\{\{0,1,2,3\},\{0,1,5,4\},\{0,3,7,4\},\{1,2,6,5\},\{2,3,7,6\},\{4,5,6,7\}\}$$
$$\{\{0,1\},\{0,3\},\{0,4\},\{1,2\},\{1,5\},\{2,3\},\{2,6\},\{3,7\},\{4,5\},\{4,7\},\{5,6\},\{6,7\}\}$$
$$\{0,1,2,3,4,5,6,7\}$$

If we identify $\{0,1,...,7\}$ with the points $\{0,0,0\},\{0,0,1\},\{0,1,0\},\{0,1,1\}$, $\{1,0,0\},\{1,0,1\},\{1,1,0\},\{1,1,1\}$ in $\mathbb{R}^3$, we could embed this as exactly the hollow unit cube. The following simplicial complexes (given in terms of their facets) are

triangulations of the Cube.

$$C_1 = \{012, 015, 023, 034, 045, 047, 125, 237, 256, 267, 457, 567\}$$
$$C_2 = \{013, 015, 034, 045, 123, 126, 156, 237, 267, 347, 457, 567\}$$

However, their critical groups are not isomorphic; $\text{Crit}(C_2) \approx \mathbb{Z}/12\mathbb{Z}$, while $\text{Crit}(C_1)$ is trivial.

Chapter 3

Oriented Matroid Theory

We are primarily concerned with oriented matroids (OMs), but we will begin with matroids.

3.1 Matroids

A Matroid, $\mathcal{M}$ is a "dependence structure," generalizing linear dependence on a set of elements, $E(\mathcal{M}) = \{e_1, ..., e_n\}$ (in this thesis, the element set is always finite). It's often convenient to assume an ordering on the element set; we say $e_i \prec e_j$ iff $i < j$.

A matroid is defined by its elements and the dependencies between them. There are numerous axiomatizations (each of which fully characterizes the matroid), we consider several: independent sets, rank functions, bases, circuits, and cocircuits.

One of the sources of inspiration for the definition of matroids is the study of the linear dependencies of the columns of a matrix. I've found matroids are best understood and explained via analogy to the matrix case, so I'll start by explaining this special case.

3.1.1 Matroids from Matrices

Every matroid, $\mathcal{M}$, has some element set, $E(\mathcal{M})$. One way of defining matroids in via their **independent sets**, some special subsets of $E(\mathcal{M})$. For any finite set of vectors, $V \subset \mathbb{R}^n$, we can define a matroid, $\mathcal{M}(V)$, with element set $E(\mathcal{M}(V)) = V$, by letting a subset of vectors be independent iff it is linearly independent. Starting with a matrix, A, we form the matroid $\mathcal{M}(A)$, from the set of column vectors of A, as just described.

The independent set axioms of matroids generalize some conditions of linear independence, as follows:

* (I0) $0 \in I$

* (I1) $X \subset Y \in I \implies X \in I$

* (I2) $X, Y \in I,\ |X| > |Y| \implies \exists x \in X \text{ s.t. } Y \cup \{x\} \in I$

We call a set of elements **dependent** iff it is not independent. We can understand the other axiomatizations in terms of independence and in analogy with the vector case. The circuits are simply the minimal dependent sets, while the bases are the maximal independent sets. The rank function assigns a number to every subset of the elements which corresponds the rank of the space they span in the vector case. The cocircuits are the circuits of the **dual matroid,** $\mathcal{M}^*$. Duality is fundamental in matroid theory, and oriented matroid theory. For every matroid, $\mathcal{M}$, we can define the dual matroid, $\mathcal{M}^*$, with the same element set,

$$E(\mathcal{M}) = E(\mathcal{M}^*),$$

and the property that

$$\mathcal{M}^{**} = \mathcal{M}.$$

A simple characterization of the dual matroid is given in terms of the bases. A set of elements, B, is a basis of $\mathcal{M}^*$ iff $B^c = E \setminus B$ is a basis of $\mathcal{M}$.

We also have that the circuits of a matroid are the cocircuits of the dual. We denote the circuits/cocircuits of a matroid by $\mathcal{C}_\mathcal{M}$ / $\mathcal{C}^*_\mathcal{M}$. We may drop the subscripts when it is unambiguous which matroid we're referring to. The duality between circuits and cocircuits can be written as follows:

$$\mathcal{C}_{\mathcal{M}^*} = \mathcal{C}^*_\mathcal{M}.$$

3.1.2 Graphic Matroids

The invention of matroids was also inspired by graph theory. For every graph, G, we define the **Cycle Matroid of** G, $\mathcal{M}(G)$, whose element set is $E(G)$ and whose dependent sets are sets of edges containing a cycle (so the circuits of the matroid are the cycles of the graph). A consequence of this construction is that the cocircuits of the matroid are the cutsets of the graph. Thus, for any graph, $\mathcal{M}^*(G)$ is called the **Cutset Matroid of** G, and its circuits are exactly the cutsets of G.

3.2 Oriented Matroids

We can modify our above procedures to create **oriented matroids (OMs)** from matrices and graphs. Oriented Matroids are like Matroids, but they encode directional information as well as dependence information. For every Oriented Matroid, $\mathcal{M}$ there is an **underlying matroid**, denoted $\underline{\mathcal{M}}$ (we then call $\mathcal{M}$ an **orientation of** $\underline{\mathcal{M}}$; not all matroids have an orientation, we call those that do **orientable**.) In general, we adopt the convention that for any signed object, the underlying object, in which we forget the sign information, is denoted with an underline.

3.3 Axiomatizations

Here we give the rank, basis, circuit and cocircuit axioms for a matroid/OM. As mentioned, each of these can be used to define an OM or a matroid. While it may seem redundant to give so many axiomatizations, we will use each of them in turn in this thesis. In general, it's not abnormal in matroid theory to translate between axiom systems frequently, and in fact many more axiom systems we will not use or discuss exist. I will refer to the "matrix case", where the matroid is taken from a matrix in the above fashion ($\mathcal{M} = \mathcal{M}(A)$, for some A,) and to graphic matroids to help explain the motivation and intuition behind some of these characterizations.

To "orient" (that is, produce an orientation of) a matroid it is necessary to attached signs (+ or -) to the bases, or to *the elements of* the circuits or cocircuits. Just as the matroid information of any of these axiomatizations determines all the others, so the sign information is entirely contained in any one of these sets, so OMs are uniquely determined by their signed bases (chirotopes), signed circuits or circuits. To encode this information, we use *signed sets*, and *chirotopes*. We identify $\{+, -\}$ and $\{+1, -1\}$ when convenient.

3.3.1 Signed Sets

Signed sets are like sets, but every element is considered to be either positive or negative. We define a **signed set**, X, to be an ordered pair, $X = \{X^+, X^-\}$. We call X^+ **the positive part of** X, its elements are the **positive members of** X. We call X^- **the negative part of** X, its elements are the **negative members of** X. The **underlying set** of a signed set, X, is $\underline{X} := X^+ \cup X^-$.

3.3.2 Vector Representation

Since all of our signed sets are subsets of the element set of an oriented matroid, $\{e_1, .., e_n\}$, we can represent a signed set as a vector in $\mathbb{Z}E$, identified with a vector in $\mathbb{Z}^n$ via this natural identification:

$$\mathbb{Z}E \longrightarrow \mathbb{Z}^n$$
$$e_i \mapsto \mathbf{e}_i$$

with e_i being the i-th matroid element, and $\mathbf{e}_i$ being the i-th standard basis vector. This allows us to talk about the components of a signed set, referencing the component of the corresponding vector in $\mathbb{Z}^n$. By forgetting about the signs of the components, we introduce the __ mapping. If we let a 1 represent inclusion of the corresponding element in the set, then this mapping gives the underlying set of a signed set. We extend __ to a mapping from OMs to matroids: simply choose a signed set representation of an OM, $\mathcal{M}$, apply __ to all of the signed sets (circuits/cocircuits), and the resulting sets are the circuits/cocircuits of $\underline{\mathcal{M}}$. This notation agrees with the earlier definition of the underlying matroid.

3.3.3 String Representation

To a signed set, X, we associate a string $\mathcal{S}^X = \mathcal{S}_1^X \mathcal{S}_2^X ... \mathcal{S}_k^X$ from the alphabet $\{e_1, ..., e_n, \bar{e}_1, ..., \bar{e}_n\}$. Since the element set is totally ordered, with the condition that the characters' subscripts form an increasing sequence, this is a bijection. For such a string, we define the argument and sign of each character:

$$\begin{aligned} \arg(e_i) &= \arg(\bar{e}_i) = e_i. \\ \text{sign}(e_j) &= \ \ 1, \ \forall\, j. \\ \text{sign}(\bar{e}_j) &= -1, \ \forall\, j. \end{aligned}$$

The total ordering also allows us to identify X^+ and X^- with strings from the alphabet $\{e_1, ..., e_n\}$ with increasing subscripts. We let $X_i^{+/-}$ be the i-th component of the string corresponding to $X^{+/-}$, considered again as an element of the matroid (not a character).

3.3.4 Example

Let X be a signed set with $X^+ = \{e_3, e_4, e_7\}$, $X^- = \{e_2, e_6\}$. Then as a vector,

$$X = \{0, -, +, +, -, 0, +\},$$

and as a string,

$$X = \bar{e}_2 e_3 e_4 \bar{e}_5 e_7$$

and referencing components,

$$\begin{aligned} &X_1 && = X_6 && = \ 0 \\ &X_2 = \text{sign}(\mathcal{S}_1^X) && = X_5 = \text{sign}(\mathcal{S}_4^X) && = -1 \\ &X_3 = \text{sign}(\mathcal{S}_2^X) && = X_4 = \text{sign}(\mathcal{S}_3^X) = X_7 = \text{sign}(\mathcal{S}_5^X) && = \ 1 \end{aligned}$$

$$\begin{aligned} X_1^+ &= \arg(\mathcal{S}_2^X) = e_2 \\ X_2^+ &= \arg(\mathcal{S}_3^X) = e_3 \\ X_3^+ &= \arg(\mathcal{S}_5^X) = e_4 \\ X_1^- &= \arg(\mathcal{S}_1^X) = e_5 \\ X_2^- &= \arg(\mathcal{S}_4^X) = e_7 \end{aligned}$$

3.3.5 Rank Axioms

The **rank** of a matroid is a function,

$$\rho : \mathbb{P}(E) \longrightarrow \{0, 1, ..., r\}$$

satisfying the following axioms:

* (R0) $\rho(X) \leq |X|$

* (R1) $X \subset Y \subset E \Longrightarrow \rho(X) \leq \rho(Y)$
* (R2) $X, Y \subset E \Longrightarrow \rho(X \cap Y) + \rho(X \cup Y) \leq \rho(X) + \rho(Y)$

In the matrix case, the rank of a set $F \subset E$, is just the dimension of the space spanned by the elements of F :

$$\rho(F) = \mathrm{Span}(F).$$

The number r, is called the **rank of** $\mathcal{M}$, and ρ has the property that:

$$\rho(B) = r \text{ iff } B \text{ contains a basis of } \mathcal{M}.$$

3.3.6 Basis/Chirotope Axioms

This brings us to the basis definition of a matroid. From the properties of the rank function, we can tell that a basis of a matrix, A, is exactly a minimal subset of the column vectors, $B \subset E(\mathcal{M}(A))$, such that $\rho(B) = r$, i.e., B is a basis for the space spanned by the columns of the matrix, $\mathrm{Im}(A)$. Every matroid corresponds uniquely (up to **loops**) to a set of bases, $\mathcal{B}$. The bases are r-element subsets of E, satisfying the following axioms:

* (B0) $\mathcal{B} \neq 0$
* (B1) $X, Y \in \mathcal{B}, X \neq Y \Longrightarrow Y \nsubseteq X,\ X \nsubseteq Y$
* (B2) $X, Y \in \mathcal{B}, X \neq Y,\ x \in X \setminus Y \Longrightarrow \exists y \in Y \setminus X \text{ s.t. } X \setminus \{x\} \cup \{y\} \in \mathcal{B}$

A **loop** is an element which is dependent as a singleton (and hence appears in no basis of $\mathcal{M}$).

In an OM, we sign the bases. We represent this by the matroid **Chirotope**,

$$\chi : \{r - \text{element subsets of } E\} \to \{-1, 0, 1\}.$$

As mentioned, $\underline{\chi}(B) = 1$ iff B is a basis of $\underline{\mathcal{M}}$, in other words, $\underline{\chi} = |\chi|$ is just the indicator function of the set $\mathcal{B}$. But χ not only says which r-element subsets are bases, it also assigns each basis an orientation, $+1$ or -1.

In the matrix case, the chirotope is just:

$$\chi(x_1, ..., x_r) = \mathrm{sign}(\det(x_1, ..., x_r)).$$

In general, the chirotope must satisfy an abstraction of the Grassmann-Plücker relations for r-order determinants. The Grassman-Plücker relations are:

$$\det(x_1, ..., x_r)\det(y_1, ..., y_r) = \sum_{i=1}^{r} \det(y_i, x_e, ..., x_r)\det(y_1, ..., y_{i-1}, x_1, y_{i+1}, ..., y_r).$$

The corresponding condition on the chirotope is:

$(B2')$ for all $x_1, ..., x_r,\ y_1, ..., y_r \in E$ such that

$$\chi(y_i, x_2, ..., x_r)\ \chi(y_1, ..., y_{i-1}, x_1, y_{i+1}, ..., y_r) \geq 0,$$

for all $i = 1, ..., r$, we have

$$\chi(x_1, ..., x_r)\ \chi(y_1, ..., y_r) \geq 0.$$

A **Chirotope** is an alternating map,

$$\hat{\chi} : E^r \longrightarrow \{-1, 0, 1\}$$

Since $\hat{\chi}$ is alternating, we know $\hat{\chi}(x) = 0$, for all $x \in E^r$ such that there exist i, j, $(i \neq j)$ with $x_i = x_j$. So we can consider $\hat{\chi}$ restricted to r-element subsets of E, and view it as attaching signs to the bases of a matroid, so that $\hat{\chi}(B) \neq 0$ iff B is a basis of $\underline{\mathcal{M}}$.

Since E is ordered, every subset is ordered canonically, so we may consider the chirotope as a map from *unordered* r-element subsets:

$$\chi : \{r\text{−element subsets of } E\} \rightarrow \{-1, 0, 1\} := \hat{\chi}\,|_{\{\{x_1, ..., x_r\}\,|\, x_i \prec x_{i+1},\, \forall\, i\}}$$

Since $\hat{\chi}$ is alternating, the mappings χ and $\hat{\chi}$ determine one another uniquely.

The underlying matroid's bases are also easily determined by the chirotope of $\mathcal{M}$. They are exactly

$$\{B \mid \underline{\chi}(B) = 1\}$$

where $\underline{\chi} := (\_ \circ \chi)$.

3.3.7 Circuit Axioms

As mentioned, the circuits of a matrix are the minimal dependent sets of vectors. The following circuit axioms are used to characterize a matroid, $\mathcal{C}(\mathcal{M}) \subset E(\mathcal{M})$:

* (C0) $0 \notin \mathcal{C}$
* (C1) $\mathcal{C} = -\mathcal{C}$
* (C2) $\forall\, X, Y \in \mathcal{C},\ \ \underline{X} \subset \underline{Y} \Rightarrow X = Y$ or $X = -Y$
* (C3) $X, Y \in \mathcal{C},\ X \neq Y,\ e \in X^+ \cap Y^- \Longrightarrow \exists Z \in \mathcal{C}$ s. t. $Z^+ \subset (X^+ \cup Y^+) \setminus \{e\}$ and $Z^- \subset (X^- \cup Y^-) \setminus \{e\}$

Because of axioms (C0) and (C1), circuits come in pairs, $\{c, -c\}$. When listing the circuits, we often list only one of each pair.

The circuits are also used to construct graphic matroids in a straightforward manner. For a graph, G, we take as elements edges of G,

$$E(\mathcal{M}) = E(G),$$

and let the circuits of $\mathcal{M}$ be the cycles (or circuits) of the graph.

3.3.8 Cocircuit Axioms

The cocircuits correspond in the graphic case to cutsets of the graph. Since they also are the circuits of the dual matroid,

$$\mathcal{C}(\mathcal{M}^*) = \mathcal{C}^*(\mathcal{M}),$$

we can conclude that (planar) graphic duality corresponds to matroid duality. In fact, more is true, a graph G is planar iff its matroid dual, $\mathcal{M}^*(G)$, is graphic.

The cocircuit axioms a matroid are easily obtained by "dualizing" the circuit axioms. We use $\mathcal{C}^*$ to refer to the set of cocircuits.

* (CC0) $0 \notin \mathcal{C}^*$
* (CC1) $\mathcal{C}^* = -\mathcal{C}^*$
* (CC2) $\forall X, Y \in \mathcal{C}^*, \ \ \underline{X} \subset \underline{Y} \Rightarrow X = Y$ or $X = -Y$
* (CC3) $X, Y \in \mathcal{C}^*, \ X \neq -Y, \ e \in X^+ \cap Y^- \Longrightarrow \exists Z \in \mathcal{C}^*$ s. t.
  $Z^+ \subset (X^+ \cup Y^+) \setminus \{e\}$ and $Z^- \subset (X^- \cup Y^-) \setminus \{e\}$

Because of axioms (CC0) and (CC1), cocircuits come in pairs, $\{c, -c\}$. When listing the cocircuits, we often list only one of each pair.

3.4 The Critical Group

Definition. *Subsuming that definition of the critical group of an undirected graph, we define the* **critical group** *of an oriented matroid* $\mathcal{M}$, *denoted* $\mathrm{Crit}(\mathcal{M})$ *to be:*

$$\mathrm{Crit}(\mathcal{M}) = \mathbb{Z}E/(\mathfrak{C} + \mathfrak{C}^*),$$

where $\mathfrak{C} = \mathrm{Span}_{\mathbb{Z}}(\mathcal{C})$, $\mathfrak{C}^* = \mathrm{Span}_{\mathbb{Z}}(\mathcal{C}^*)$, are the **circuit space** and **cocircuit space**, respectively.

A trivial consequence of this definition is that

$$\mathrm{Crit}(\mathcal{M}) = \mathrm{Crit}(\mathcal{M}^*).$$

To find the critical group of an undirected graph, we first give each edge an orientation. As the critical group of an undirected graph is well-defined, it follows that it is independent of the choice of orientation. This leads us to ask the following question:

Question. *Is it the case that for any orientable matroid, $\mathcal{M}$, all orientations of $\mathcal{M}$ have the same critical group?*

We know the answer is yes in the graphic and uniform cases. The graphic case is well known, we prove the uniform case in Theorem 3.1.

3.5 The Sandpile Group

In graphs, the vertices are not represented explicitly in the matroid. But the sandpile group is defined using the vertices. It turns out that in a graph, the **fundamental cocircuits** of the matroid correspond perfectly to singleton vertex cuts. Suho Oh has suggested [20] that fundamental cocircuits might serve as a vertex analog for construction of the matroid sandpile group.

For a matroid $\mathcal{M}$, we let $\mathcal{F}_{\mathcal{M}}(B,x) = \mathcal{F}(B,x)$ denote **the fundamental circuit of the basis** B **with respect to an element** $x \notin B$. This is the unique circuit of $\mathcal{M}$, $C \in \mathcal{C}$ contained in $B \cup \{x\}$ and containing x :

$$x \in \mathcal{F}(B,x) \subset B \cup \{x\}.$$

$\mathcal{F}^*_{\mathcal{M}}(B,y) = \mathcal{F}^*(B,y)$ denotes **the fundamental cocircuit of** B **with respect to** $y \in B$. This is the unique cocircuit disjoint from $B \setminus \{y\}$, and containing y :

$$y \in \mathcal{F}^*(B,y) \subset (B \setminus \{y\})^c.$$

The fundamental circuits and cocircuits are related by

$$\mathcal{F}^*_{\mathcal{M}}(B,y) = \mathcal{F}_{\mathcal{M}^*}(E \setminus B, y).$$

3.6 Equivalence for Oriented Matroids

We already have one equivalence relation on the class of oriented matroids: we say that $\mathcal{M}$ and $\mathcal{M}'$ are **reorientation equivalent** iff $\underline{\mathcal{M}} = \underline{\mathcal{M}'}$. Now, we define an isomorphism of oriented matroids, $\mathcal{M}$ and $\mathcal{M}'$. This is a bijection, $f : E(\mathcal{M}) \longrightarrow E(\mathcal{M}')$, such that the matroid structure is preserved. Since all the axiomatizatons are equivalent, we only need prove that one of the circuits, cocircuits, chirotope, etc., are preserved by f. It is easy to see that the critical group is invariant under isomorphism. This result is preserved even if we take the more general notion of isomorphism from [24], which doesn't require a bijection between the element sets, allowing for "insertion or deletion" of **parallel elements** and **loops**. Elements e, f are **parallel** if $\{e, f\}$ is a circuit. An element e is a loop if $\{e\}$ is a circuit. Where we have resigning, [24] instead has relabeling, allowing these insertions and deletions.

If we consider OMs on some fixed element set, E, it makes sense to distinguish isomorphism from equality. We will define two more equivalence relations on the set of oriented matroids (not their isomorphism classes) with a given element set, E. We define their operation on elements and extend linearly to signed set representations.

Let $\mathcal{O}_n$ be the set of oriented matroids with element set $E = \{e_i, ..., e_n\}$. Define the **resigning map,** resign : $E \times \mathcal{O}_n \longrightarrow E$, by

$$\begin{aligned} \text{resign} : E \times \mathbb{Z}E &\longrightarrow \mathbb{Z}E \\ (e_i, e_j) &\mapsto (-1)^{\delta_{ij}} e_j \end{aligned}$$

and extending linearly. So the resign map just switches the sign on the e_i-th coordinate of a vector $x \in \mathbb{Z}E$. This induces a resign map on OMs, to resign an OM, $\mathcal{M}$, we choose any representation of $\mathcal{M}$ as a set of sign vectors in $\mathbb{Z}E$, apply the resign map to this set, and the resulting sign vectors determine an OM.

We say $\mathcal{M}$ and $\mathcal{M}'$ are **resigning equivalent** iff there is some $F = \{f_1, .., f_k\} \subset E$ such that resigning $\mathcal{M}$ by $\{f_1, .., f_k\}$ in succession gives $\mathcal{M}'$. In other words,

$$\text{resign}(F, \mathcal{M}) = \mathcal{M}',$$

where we have extended the resign as just described.

We define the action of the symmetric group, S_n on an OM via permutation of the element set:

$$\begin{aligned} \text{perm} : S_n \times E &\longrightarrow E \\ \{\sigma, e_i\} &\mapsto e_{\sigma(i)}. \end{aligned}$$

We extend this via linearity to a map perm : $S_n \times \mathbb{Z}E \longrightarrow \mathbb{Z}E$, and this map induces an operation on OMs with n elements for each permutation. We simply apply perm to any signed set representation of our OM, considered as a vector in $\mathbb{Z}E$, and the result is another such OM on the same element set. We call two OMs, $\mathcal{M}$ and $\mathcal{M}'$, **permutation equivalent** iff there exists a $\sigma \in S_n$ such that $\text{perm}(p, \mathcal{M}) = \mathcal{M}'$.

If they are well-defined, it is easy to show that resign : $\mathbb{P}E \times \mathcal{O}_n \longrightarrow \mathcal{O}_n$ and $\text{perm}_n : S_n \times \mathcal{O}_n \longrightarrow \mathcal{O}_n$ define bonified transitive, symmetric, and reflexive equivalence relations:

Reflexivity:

$$\begin{aligned} \text{resign}(\emptyset, \mathcal{M}) &= \mathcal{M} \\ \text{perm}(\emptyset, \mathcal{M}) &= \mathcal{M}. \end{aligned}$$

Symmetry:

$$\begin{aligned} \text{resign}(F, \mathcal{M}) = \mathcal{M}' &\Longrightarrow \text{resign}(F, \mathcal{M}') = \mathcal{M}. \\ \text{perm}(\sigma, \mathcal{M}) = \mathcal{M}' &\Longrightarrow \text{perm}(\sigma, \mathcal{M}') = \mathcal{M}. \end{aligned}$$

Transitivity:

$$\begin{aligned} \text{resign}(F, \mathcal{M}) = \mathcal{M}',\ \text{resign}(G, \mathcal{M}') = \mathcal{M}'' &\Longrightarrow \text{resign}(F \cup G \setminus (F \cap G), \mathcal{M}) = \mathcal{M}''. \\ \text{perm}(\sigma, \mathcal{M}) = \mathcal{M}',\ \text{perm}(\sigma', \mathcal{M}') = \mathcal{M}'' &\Longrightarrow \text{perm}(\sigma'\sigma, \mathcal{M}) = \mathcal{M}''. \end{aligned}$$

It must be verified that resign : $\mathbb{P}E \times \mathcal{O}_n \longrightarrow \mathcal{O}_n$ and $\text{perm}_n : S_n \times \mathcal{O}_n \longrightarrow \mathcal{O}_n$ are in fact well-defined. Since they were defined by linear extension on basis elements, we know the mappings on signed sets, resign : $\mathbb{P}E \times \mathbb{Z}E \longrightarrow \mathbb{Z}E$ and perm : $S_n \times \mathbb{Z}E \longrightarrow \mathbb{Z}E$, are well-defined, the question is whether any we can apply these mappings to any representation of an OM, $\mathcal{M}$ and get the same OM, $\mathcal{M}'$, out, independent of choice of representation. The answer is yes. We leave this to the reader.

We also would like to know the effect of resigning or permuting on the chirotope of a matroid. We have that

$$\text{resign}(e, \chi)(B) = \left\{ \begin{matrix} \chi(B) \text{ if } e \notin B \\ -\chi(B) \text{ if } e \in B. \end{matrix} \right\}.$$

for any $e \in E$, and any r-element $B \subset E$. Hence, resigning by a set, $F \subset E$, gives:

$$\text{resign}(F, \chi)(B) = (-1)^{|F \cap B|} B$$

This is an easy consequence of the cocircuit/chirotope translation given in [16].

For a permutation, σ, we just have

$$\text{perm}(\sigma, \chi)(B) = \chi(\sigma(B)).$$

3.7 Oriented Matroids from Point Configurations

We use $(x_1, ..., x_n)$ for coordinates of the vector $\vec{x}$. For a set of vectors, we use $\{\mathbf{x}_1, ..., \mathbf{x}_k\}$. Ziegler's *Lectures on Polytopes* [14] was the primary reference for this section.

Every finite set of vectors, $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ spanning an r-dimensional vector space gives rise to an oriented matroid with circuits given by the minimal linear dependencies of the vectors in the following way: for each such dependence,

$$\sum_{i=1}^{k} \lambda_i \mathbf{v}_i = 0$$

we associate the sign vector

$$D = (\text{sign}(\lambda_1), ..., \text{sign}(\lambda_k)).$$

Define the **support of D** as the set $\{i : D_i \neq 0\}$. The circuits of the oriented matroid are then the sign vectors with minimal support.

Every affine point configuration

$$X = \{p_1, ..., p_k\} \subset \mathbb{R}^n$$

also gives rise to an associated oriented matroid via an associated set of vectors, which we will call the **projectivization of X**, denoted by

$$\text{proj}(X) := \{\mathbf{v}_1, ..., \mathbf{v}_k\} \subset \mathbb{R}^{n+1}, \qquad \mathbf{v}_i = (1, p_i), \ \forall i.$$

The cocircuits of a point configuration correspond to hyperplanes passing through some of the points. Each such hyperplane divides the set of points into the points on it, the points "above it" and the points "below" it (once we decide which way is "up"). These translate into sign vectors, with + for the points above, - for those below, and 0 for those on the hyperplane. A cocircuit is then one such sign vector with minimal support (given by a hyperplane passing through a maximal set of the points).

3.8 Uniform Oriented Matroids

An (oriented) matroid $\mathcal{M}$ is called **k-uniform**, or simply **uniform**, if its bases are exactly the k-element subsets of $E(\mathcal{M})$.

We were able to completely classify the critical groups of uniform oriented matroids. Let $U(n, r)$ denote the uniform matroid of rank r on n elements. We have the following result:

Theorem 3.1. *The critical group of a uniform oriented matroid is independent of orientation. For any orientation of $U(n, r)$, we have:*

$$\mathrm{Crit}(U(n,r)) \approx \begin{cases} \mathbb{Z}/n\mathbb{Z} : & r \in \{1, n-1\} \\ \mathbb{Z}/2\mathbb{Z} : & r \text{ odd, } n \text{ even}, 2 < r < n-2 \\ 0 \quad : & \text{otherwise.} \end{cases}$$

Proof. The following proof uses results from [22] and OMs constructed from point configurations (and one that isn't). Hoffstätler and Nickel[22] give the dimension and in some cases an explicit formulation of the circuit space of an OM, $\mathfrak{C}(\mathcal{M})$ (which he calls the **circuit lattice of** $\mathcal{M}$, denoted $\mathcal{F}(\mathcal{M})$.) Instead of letting points, p be the elements of the matroid, we will make corresponding matroid elements, e_p. This allows us to disambiguate between addition/subtraction of points as points, and addition/subtraction of sign vectors corresponding to matroid elements. Note that we still use $\mathbf{e}_i$ to denote standard basis vectors, so $\mathbf{e}_i \neq e_i$.

First, suppose $r = 1$. Then the bases of $\mathcal{M}$ are just the singleton subsets of E. Since the critical group is invariant under resigning of elements, all rank 1 OMs are in the same resigning class and hence have the same critical group. Now, consider the set of points

$$X = \{0, \mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_{n-2}, p\} \subset \mathbb{R}^{n-2},$$

where

$$p = \sum_{i=1}^{n-2} \lambda_i \mathbf{e}_i, \quad \sum_{i=1}^{n-2} \lambda_i < 1, \text{ and } \lambda_i > 0.$$

These points constitute the points of the $(n-2)$-dimensional simplex with the point p somewhere in the convex hull of the other points. This point set determines a uniform OM of rank $n-1$. By the affine dependence definition of a circuit, we know that the only circuit of the OM is $(+, +, ..., +, -)$, since any proper subset of E is linearly independent, and p is defined as a positive linear combination of the $\mathbf{e}_i$. Any $n-2$ points lie on a hyperplane in $n-2$ dimensional space, so the cocircuits can be found by looking at all $n-2$ element subsets of E. Let $H \subset E$, with $|H \cap X| = n-2$. Let $H^c \cap X = \{x, y\}$.

Suppose $p \notin H^c$. Then H cuts the simplex in half with x and y on opposite sides, so the resulting cocircuit is $\pm(e_x - e_y)$. On the other hand, if $p \in H^c$, then H is one of the facets of the simplex, so x and y are on the same side of H, and the resulting cocircuit is $\pm(e_x + e_y)$.

By setting cocircuits equivalent to 0, we find that $e_p + e_x = 0,\ \forall\ e_x \in E,\ x \neq p$. By also setting the circuit equivalent to 0, we find that $e_p = (n-1) * e_x,\ \forall\ e_x \in E,\ x \neq p$, implying $n * p = 0$. The critical group is exactly $\mathbb{Z}E$ modulo these equivalence relations, i.e., $\mathbb{Z}/n\mathbb{Z}$.

Since there is only one isomorphism class for rank 1 (and hence rank $n-1$) matroids, this is the critical group for any uniform matroid on n elements with rank 1 or $n-1$.

Now, suppose the second condition holds, so we know r is odd, n is even, and $2 < r < n-2$. We use several results from (H&N). First, we have that $\mathfrak{C}$ is either the set of things whose coefficients sum to an even number,

$$\mathfrak{C} = \{x \in \mathbb{Z}E | \textstyle\sum_{e \in E} x_e = 0(2)\},$$

where x_e is of course, the e-th component of the vector $x \in \mathbb{Z}E$, or $\mathfrak{C}$ has dimension $(n-1)$. Furthermore, a further result gives that all those OMs with $\dim(\mathfrak{C}) = n-1$, are in the same isomorphism class, so have the same critical group.

In the case that $\mathfrak{C} = \{x \in \mathbb{Z}E | \sum_{e \in E} x_e = 0(2)\}$, the critical group is either $\mathbb{Z}/2\mathbb{Z}$, or trivial, since $\mathbb{Z}E/\mathfrak{C} \approx \mathbb{Z}/2\mathbb{Z}$, already, and to construct the critical group we just mod out by the cocircuits as well. But since every $r+1$ element subsets of E is dependent and thus a circuit of $U(n, r)$, by duality, the cocircuits are the $n-r+1$ element subsets of E. So when n is even and r is odd, every cocircuit has an even number, $(n-r+1)$, of non-zero entries (when considered as a sign vector). Considering the sign vector as an element of $\mathbb{Z}E$, every non-zero entry is either $+1$ or -1, and hence is equivalent to $1(2)$. Hence the sum of the coefficients of the image of the cocircuit is even, so $\mathfrak{C}^* \subset \mathfrak{C}$. This completes the proof for this case.

For the case where $\dim(\mathfrak{C}) = n-1$, I define an oriented matroid, $\mathcal{M}$, prove it is in this isomorphism class, and prove is has critical group $\mathbb{Z}/2\mathbb{Z}$.

Let $E(\mathcal{M}) = \{e_1, ..., e_n\}$, with $e_i \prec e_j$ iff $i < j$, as per usual. Let the circuit signature (up to sign) for any $X \in \mathcal{C}$ be given as follows. Let $\underline{X} = \{x_1, ..., x_{2k}\}$, with $x_i \prec x_{i+1},\ \forall\ i$. Then define $X^+ = \{x_i \mid i = 1(2)\}$, $X^- = \{x_i \mid i = 0(2)\}$. We must first show that the circuits thus signed obey the axioms. In particular, we must show that:

$$\forall\, X, Y \in \mathcal{C}, X \neq -Y, \quad \forall\, e \in X^+ \cap Y^-, \quad \exists\, Z \in \mathcal{C} \text{ s.t.}$$
$$Z^+ \subset (X^+ \cup Y^+) \setminus \{e\} \text{ and } Z^- \subset (X^- \cup Y^-) \setminus \{e\}$$

Let X, Y and e satisfy the above. Notice that with the above notation, $x_i = \arg(\mathcal{S}_i^X)$, and $y_i = \arg(\mathcal{S}_i^Y)$, for all i, so then $e = x_i = y_j$, for some i, j with i odd and j even. Suppose $i < j$. Then let

$$\mathcal{S}^Z = \{\mathcal{S}_i^Y \mid i < j\} \centerdot \{\mathcal{S}_i^X \mid i \geq j\},$$

with the dot, $\centerdot$, representing concatenation, so the elements of Z inherit their signs from X and Y, so $z \in Z^+$ iff $z \in X^+ \cup Y^+$.

Notice that $\underline{Z}$ contains exactly $j-1$ elements of $\underline{Y}$ and $r-j+2$ elements of $\underline{X}$, for a total of $r+1$ elements. Thus $\underline{Z}$ is a circuit of the matroid, and hence a signed circuit; we only took elements greater than e from X and only took elements less than e from Y, and hence $\arg(\mathcal{S}_i^Z) \prec \arg(\mathcal{S}_{i+1}^Z)$, for all i, and hence the signs of the elements of Z are consistent with the circuit signature. Finally, since $\underline{Z}$ doesn't contain e, we have proven the axiom holds, for the case $i < j$.

For the case that $i > j$, simply consider $X' = -Y$, $Y' = -X$. Then we have $e = x_i' = y_j'$, with $i < j$, and can apply the result just obtained on X', Y' to produce a Z'. Observe that $Z' = -Z$, and hence, by axiom (C1), $Z \in \mathcal{C}$.

We are left with the case that $i = j$, in which case, by the way we defined the circuit signature, $\text{sign}(\mathcal{S}_i^X) = \text{sign}(\mathcal{S}_i^Y)$, for all i. Now, since $X \neq -Y$, we know $\underline{X} \neq \underline{Y}$, and consequently, $\exists\, b \text{ s. t. } x_b \neq y_b$. We want a to be the spot closest to i where X and Y differ. Let a be the maximal such b less than i, if one exists, otherwise, let a be the *minimal* such b *greater* than i. Notice that as defined, $a < i$ implies $x_{a+1} = y_{a+1}$, and $a > i$ implies $x_{a-1} = y_{a-1}$. Now, there are four conditions:

- If $a < i$, and $x_a < y_a$, then $\underline{Z} = \underline{X} \cup \{y_a\} \setminus \{x_{a+1}\}$ defines a circuit.
- If $a > i$, and $x_a > y_a$, then $\underline{Z} = \underline{X} \cup \{y_a\} \setminus \{x_{a-1}\}$ defines a circuit.
- If $a < i$, and $x_a > y_a$, then $\underline{Z} = \underline{Y} \cup \{x_a\} \setminus \{y_{a+1}\}$ defines a circuit.
- If $a > i$, and $x_a < y_a$, then $\underline{Z} = \underline{Y} \cup \{x_a\} \setminus \{y_{a-1}\}$ defines a circuit.

These can all be verified very easily. We have guaranteed that the element to be inserted is in order and has the correct sign.

Now, observe that the coefficient sum of the image of any circuit of $X \in \mathcal{C}$ is 0. Thus using (H&N), $\dim(\mathfrak{C}) = n-1$, and so this matroid is from the isomorphism class of interest. But we also want to show that all of the elements of the matroid are in fact equivalent when we quotient by $\mathfrak{C}$. By assumption, $r < n-2$, thus the following are circuits of $\underline{\mathcal{M}}$:

$$\underline{c}_1 = \{e_1, ..., e_{i-1}, e_i\ , e_{i+2}, ..., e_{r+2}\}$$
$$\underline{c}_2 = \{e_1, ..., e_{i-1}, e_{i+1}, e_{i+2}, ..., e_{r+2}\}$$

and $c_1 - c_2 = e_i e_{i+1}$. Thus $e_i \sim e_{i+1}$, $\forall\, i$.

Now, we give a cocircuit with image -2. Since all the cocircuits have even image, this will finish the proof. Define $d \in \mathcal{C}^*$, by

$$d^+ = \{e_1, e_2, ..., e_{(n-r-1)/2}\}$$
$$d^- = \{e_{(n-r+3)/2}, e_{(n-r+5)/2}, ..., e_{n-r+2}\}$$

Then $|\underline{d}| = n - r + 1$, so to prove $d \in \mathcal{C}^*$, it suffices to show that $d \perp c$, $\forall\ c \in \mathcal{C}$. So lets suppose $c \in \mathcal{C}$, but c and d are not perpendicular. Suppose $|\underline{c} \cap d^+| > 1$ or $|\underline{c} \cap d^-| > 1$. Then there exist i, j, with $i < j$, such that $\arg(\mathcal{S}^c_i), \arg(\mathcal{S}^c_j) \in d^+$, or $\arg(\mathcal{S}^c_i), \arg(\mathcal{S}^c_j) \in d^-$, so then

$$\arg(\mathcal{S}^c_{i+1}) \leq \arg(\mathcal{S}^c_j), \text{ so } \mathcal{S}^c_{i+1} \in d^+$$

or

$$\arg(\mathcal{S}^c_{j-1}) \geq \arg(\mathcal{S}^c_i), \text{ so } \mathcal{S}^c_{j-1} \in d^-.$$

But then $c \perp d$, since

$$\operatorname{sign}(\mathcal{S}^d_i)\operatorname{sign}(\mathcal{S}^c_i) = -\operatorname{sign}(\mathcal{S}^d_{i+1})\operatorname{sign}(\mathcal{S}^c_{i+1})$$

or

$$\operatorname{sign}(\mathcal{S}^d_j)\operatorname{sign}(\mathcal{S}^c_j) = -\operatorname{sign}(\mathcal{S}^d_{j-1})\operatorname{sign}(\mathcal{S}^c_{j-1})$$

Thus $|\underline{c} \cap d^-| \leq 1$, and $|\underline{c} \cap d^+| \leq 1$, and so equality necessarily holds for both, since $|\underline{c}| = r + 1$, $|\underline{d}| = n - r + 1 \Longrightarrow |\underline{c} \cap \underline{d}| \geq 2$. So now we have $\underline{c} = d^c \cup \{e, f\}$, with $e \in d^+$, $f \in d^-$. But then let $\arg(\mathcal{S}^c_i) = e$. Then necessarily, $\arg(\mathcal{S}^c_{i+1}) = (n-r+1)/2$, since $(n - r + 1)/2 \notin \underline{d}$, and $\arg(\mathcal{S}^c_{i+2}) = f$, so $\operatorname{sign}(\mathcal{S}^d_i)\operatorname{sign}(\mathcal{S}^c_i) = -\operatorname{sign}(\mathcal{S}^d_{i+2})\operatorname{sign}(\mathcal{S}^c_{i+2})$, since $\operatorname{sign}(\mathcal{S}^c_i) = \operatorname{sign}(\mathcal{S}^c_{i+2})$.

Now, suppose that neither of the first two conditions hold, so the theorem states that the critical group is 0. We consider three cases:

$$\begin{array}{rcr} (1) & : & r \in \{0, n\} \\ (2) & : & r \in \{2, n-2\},\ n \neq 3 \\ (3) & : & n\,\text{odd},\ 2 < r < n-2. \end{array}$$

For case (1), suppose $r = 0$. Then the circuits are exactly the 1 element subsets of E, thus $\mathfrak{C} = \mathbb{Z}E$.

For case (2), suppose $r = 2$. Then by Table 1 in [22], $\mathfrak{C} = \mathbb{Z}E$.

For case (3), we use a result from [22], which states that for even $r : 2 < r < n-2$, $\mathfrak{C} = \mathbb{Z}E$. By duality, then, for any $r : 2 < r < n - 2$, one of $\{r, n - r\}$ will be even, since n is odd. Thus either $\mathfrak{C} = \mathbb{Z}E$, or $\mathfrak{C}^* = \mathbb{Z}E$.

□

3.9 The Free Join

The *free join* is a construction defined in [28]. The definition below is a modification convenient for proving later theorems. I'm using the bowtie, $\bowtie$ for my definition, even though it is different from [28]'s.

Definition. *For any two point configurations,* $X = \{x_1, ..., x_a\} \subset \mathbb{R}^n$ *and* $Y = \{y_1, ..., y_b\} \subset \mathbb{R}^m$*, we define the* **canonical free join** *of* X *and* Y*, a subset of* $\mathbb{R}^{m+n+1}$ *by*

$$X \bowtie Y := \iota_X(X) \cup \iota_Y(Y),$$

where

$$\begin{aligned} \iota_X : \mathbb{R}^m &\longrightarrow \mathbb{R}^{m+n+1} \\ (x_1, ...x_m) &\mapsto (x_1, ..., x_m, 0, ..., 0) \end{aligned} \qquad \begin{aligned} \iota_Y : \mathbb{R}^n &\longrightarrow \mathbb{R}^{m+n+1} \\ (y_1, ...y_n) &\mapsto (0, ..., 0, y_1, ..., y_n, 1). \end{aligned}$$

3.9.1 What Groups Can be Constructed This Way?

It turns out that any finite abelian group is the critical group of some point configuration. We prove this below.

Lemma 3.1. *For any two point configurations,* X *and* Y*, let*

$$\text{proj}(X) = \{\mathbf{x}_1, ..., \mathbf{x}_a\} \subset \mathbb{R}^n, \quad \text{proj}(Y) = \{\mathbf{y}_1, ..., \mathbf{y}_b\} \subset \mathbb{R}^m,$$

and let $\{p_1, ..., p_a, p_{a+1}, ..., p_{a+b}\}$*, be the points of* X *and* Y *embedded in* $\mathbb{R}^{n+m+1}$ *via the canonical free join, and* $\{\mathbf{v}_1, ..., \mathbf{v}_{a+b}\} = \{(1, p_1), ..., (1, p_{a+b})\}$*, be the corresponding vectors. Let*

$$\mathcal{C}_X = \{c_1^X, ..., c_\alpha^X\}, \quad \mathcal{C}_Y = \{c_1^Y, ..., c_\beta^Y\}$$

be the circuits of $X \subset \mathbb{R}^n$ *and* $Y \subset \mathbb{R}^m$*, and let*

$$\mathcal{C}_x = \{c_1^x, ..., c_\alpha^x\},\ \mathcal{C}_y = \{c_1^y, ..., c_\beta^y\} \subset \{-, 0, +\}^{a+b}$$

be the corresponding sign vectors for the embedded points, $\{p_1, ..., p_{a+b}\}$*,*

$$c_i^x = (c_i^X, \vec{0}_b), c_j^y = (\vec{0}_a, c_j^Y).$$

Then the circuits of $X \bowtie Y$ *are exactly the sign vectors* $\mathcal{C}_x \cup \mathcal{C}_y$*.*

Proof. To each c_i^X (respectively, c_j^Y) there corresponds at least one vector,

$$\lambda^i = (\lambda_1^i, ..., \lambda_a^i) \quad \big(\text{respectively, } \gamma^j = (\gamma_1^j, ..., \gamma_b^j)\big)$$

such that

$$(\text{sign}(\lambda_1^i), ..., \text{sign}(\lambda_a^i)) = c_i^X \quad \big(\text{respectively,}(\text{sign}(\gamma_1^j), ..., \text{sign}(\gamma_b^j)) = c_j^Y\big)$$

and

$$\sum_{k=1}^{a} \lambda_k^i \mathbf{x}_k = 0 \quad (\text{respectively, } \sum_{k=1}^{b} \gamma_k^j \mathbf{y}_k = 0).$$

Then the vectors $\Lambda^i = (\lambda^i, \vec{0}_b)$ (respectively, $\Gamma^j = (\vec{0}_a, \gamma^j)$), have the property

$$\sum_{k=1}^{a+b} \Lambda_k^i \mathbf{v}_k = 0, \quad (\text{respectively, } \sum_{k=1}^{a+b} \Gamma_k^j \mathbf{v}_k = 0),$$

since

$$\sum_{k=1}^{a+b} \Lambda_k^i \mathbf{v}_k = \sum_{k=1}^{a} \lambda_k^i \mathbf{x}_k + \sum_{k=a+1}^{a+b} 0 = 0 + 0 = 0,$$

and similarly for the Γ^j's.

So then each such Λ^i, and each Γ^j give a dependence among the vectors $(\mathbf{v}_1, .., \mathbf{v}_{a+b})$, and furthermore, the corresponding sign vectors have minimal support, since otherwise the corresponding sign vectors c_i^X, c_j^Y would not have minimal support, and hence would not be circuits. Furthermore, the sign vectors of these Λ^i, Γ^j are exactly the sign vectors c_i^x, c_j^y, as desired. □

Lemma 3.2. *For any two point configurations, X and Y, let*

$$\mathcal{C}_X^* = \{d_1^X, ..., d_\alpha^X\}, \quad \mathcal{C}_Y^* = \{d_1^Y, ..., d_\beta^Y\}$$

be the cocircuits of $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$, and let

$$\mathcal{C}_x^* = \{d_1^x, ..., d_\alpha^x\}, \mathcal{C}_y^* = \{d_1^y, ..., d_\beta^y\} \subset \{-, 0, +\}^{a+b}$$

be the corresponding sign vectors for the embedded points, $X \bowtie Y = (p^1, ..., p^{a+b})$,

$$d_i^x = (d_i^X, \vec{0}_b), d_j^y = (\vec{0}_a, d_j^Y).$$

Then the cocircuits of $X \bowtie Y$ are exactly the sign vectors $\mathcal{C}_x^* \cup \mathcal{C}_y^*$.

Proof. We'll prove this using hyperplanes. Every cocircuit has a corresponding hyperplane, which can be defined by a normal vector perpendicular to the hyperplane, and a scalar representing the distance of the plane from the origin. So for each d_i^X, we take a corresponding normal vector, $S^i = (s_1^i, ..., s_n^i)$, and a scalar $\mathfrak{S}^i$. And for every d_j^Y, we take a corresponding normal vector, $T^j = (t_1^i, ..., t_m^i)$, and scalar $\mathfrak{T}^j$.

The points of the hyperplane corresponding to the vector S^i are exactly the points p such that

$$\sum_{k=1}^{n} s_k^j p_k = \mathfrak{S}^i$$

and similarly for the vectors T^j.

Thus, all the points $P = (p, q)$, where $q \subset \mathbb{R}^{m+1}$ lie on the hyperplane defined by the vector $(S^i, \vec{0}_{m+1})$, and the scalar $\mathfrak{S}^i$. This hyperplane defines a cocircuit in $X \bowtie Y$, since if it it did not have minimal support, than neither would the corresponding cocircuit, d_i^X of X.

The argument works for the cocircuits of Y as well, if we take the vector corresponding to T^j to be $(\vec{0}_n, T^j, 0)$.

It remains to show that there are no new cocircuits of $X \bowtie Y$, besides these. Suppose there were another cocircuit. Consider the corresponding hyperplane, $\mathcal{H}$, and let

$$U = (u_1, ..., u_{n+m+1})$$

be a vector defining this hyperplane. Then consider the hyperplanes in $\mathbb{R}^n$ and $\mathbb{R}^m$ defined by the vectors

$$U_X = (u_1, .., u_n), \quad U_Y = (u_{n+1}, ...u_{n+m}).$$

If either of these does not contain at least all the points contained by one of the cocircuits of X or Y, then the U does not define a cocircuit, because we can change one of U_x, or U_Y to be exactly one of the S^i or T^j which contains all of the points of $\mathcal{H} \cap \iota_X(X)$ or $\mathcal{H} \cap \iota_Y(Y)$, and at least one more.

Thus $\mathcal{H}$ must at least contain at least all the points $\iota_X(d_i^X) \cup \iota_Y(d_j^Y)$, for some i, j. Suppose it contains another point of $X \bowtie Y$. Then either U_X or U_Y does not define a hyperplane in X or Y. Since every non-zero vector defines a hyperplane, U_X, or U_Y must be 0, but not both, since then $U = (0, ..., 0, u_{n+m+1})$ must contain a point of $\iota_Y(Y)$, but this implies $u_{n+m+1} = 0$, since every point of $\iota_Y(Y)$ has a 1 in the last coordinate. And if $u_{m+n+1} = 0$, then $U = \vec{0}$, and so does not define a hyperplane. Thus any cocircuit of $X \bowtie Y$ has the specified form. □

Theorem 3.2. *For any two point configurations, X and Y,*

$$\mathrm{Crit}(X \bowtie Y) \approx \mathrm{Crit}(X) \otimes \mathrm{Crit}(Y).$$

Proof. We rewrite the statement as

$$\mathbb{Z}(X \bowtie Y)/(\mathfrak{C}_{X\bowtie Y} + \mathfrak{C}^*_{X\bowtie Y}) \approx \mathbb{Z}X/(\mathfrak{C}_X + \mathfrak{C}^*_X) \otimes \mathbb{Z}Y/(\mathfrak{C}_Y + \mathfrak{C}^*_Y),$$

where (abusing notation) $\mathfrak{C}_X$ is taken to mean the circuit space of the oriented matroid of the point configuration X, and so forth.

Now, we also have

$$\mathbb{Z}(X \bowtie Y) = \mathbb{Z}(\iota_X(X)) \otimes \mathbb{Z}(\iota_Y(Y)),$$

so we can rewrite the left side of the equation as

$$\begin{aligned}\frac{\mathbb{Z}(\iota_X(X)) \otimes \mathbb{Z}(\iota_Y(Y))}{\mathfrak{C}_{X\bowtie Y} + \mathfrak{C}^*_{X\bowtie Y}} &= \frac{\mathbb{Z}(\iota_X(X)) \otimes \mathbb{Z}(\iota_Y(Y))}{\mathfrak{C}_x + \mathfrak{C}^*_x + \mathfrak{C}_y + \mathfrak{C}^*_y} \\ &= \frac{\mathbb{Z}(\iota_X(X))}{\mathfrak{C}_x + \mathfrak{C}^*_x + \mathfrak{C}_y + \mathfrak{C}^*_y} \otimes \frac{\mathbb{Z}(\iota_Y(Y))}{\mathfrak{C}_x + \mathfrak{C}^*_x + \mathfrak{C}_y + \mathfrak{C}^*_y} \\ &= \frac{\mathbb{Z}(\iota_X(X))}{\mathfrak{C}_x + \mathfrak{C}^*_x} \otimes \frac{\mathbb{Z}(\iota_Y(Y))}{\mathfrak{C}_y + \mathfrak{C}^*_y},\end{aligned}$$

the last equality holding because all the points of $\mathbb{Z}(\iota_X(X))$ already have 0's where the elements of $\mathfrak{C}_y, \mathfrak{C}^*_y$ do not, and all the points of $\mathbb{Z}(\iota_Y(Y))$ have 0's where the elements of $\mathfrak{C}_x, \mathfrak{C}^*_x$ do not.

Now we simply observe that

$$\frac{\mathbb{Z}(\iota_X(X))}{\mathfrak{C}_x + \mathfrak{C}^*_x} \approx \mathbb{Z}X/(\mathfrak{C}_X + \mathfrak{C}^*_X)$$

via the mapping that forgets the last $m+1$ coordinates, and the mapping that puts 0's there, and similarly,

$$\frac{\mathbb{Z}(\iota_Y(Y))}{\mathfrak{C}_y + \mathfrak{C}_y^*} \approx \mathbb{Z}Y/(\mathfrak{C}_Y + \mathfrak{C}_Y^*)$$

via the mappings that forgets the first n coordinates and the last coordinate, and the mapping that puts 0's and a 1 in those coordinates. This demonstrates the claimed isomorphism. □

Corollary. *Any Finite Abelian Group is the Critical Group of some point configuration.*

Proof. By Lemma 3.1/3.2 and Theorem 3.1, the group $(\mathbb{Z}/2\mathbb{Z})^k \otimes \mathbb{Z}/n_1\mathbb{Z} \otimes \mathbb{Z}/n_2\mathbb{Z} ... \otimes \mathbb{Z}/n_l\mathbb{Z}$ is isomorphic to $\text{Crit}(Y^k \bowtie X_1 \bowtie ... \bowtie X_l)$, with Y^k the canonical free join of Y with itself k times, and Y is some point configuration representing $U(6,3)$, for instance, $\{(1,1),(1,0),(0,1),(-1,-1),(-1,0),(0,-1)\}$ (other examples abound in [17], in fact, exactly the "non-trivial" order types they list had critical group $\mathbb{Z}/2\mathbb{Z}$)

$$X_i = \{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_{n_i - 2}, p\} \subset \mathbb{R}^{n_i - 2},$$

where

$$p = \sum_{i=1}^{k} \lambda_i \mathbf{e}_i, \quad \sum_{i=1}^{k} \lambda_i < 1, \text{ and } \lambda_i > 0 \,\forall\, i.$$

□

Chapter 4

Matroid Complexes, h-vectors, and the Tutte Polynomial

In this chapter, I examine the relationship between the Tutte Polynomial, the h-vector, the number of bases, and the order of the critical group of a matroid.

4.1 The Tutte Polynomial

The Tutte Polynomial, $T(x, y)$, is best known as a graph invariant that is both hugely informative and computationally difficult. But it also generalizes perfectly well to matroids. For a matroid, $\mathcal{M}$, we define:

$$T_{\mathcal{M}}(x, y) = \sum_{S \subseteq E} (x-1)^{r(\mathcal{M})-r(S)} (y-1)^{|S|-r(S)}.$$

This definition easily captures the duality property of the Tutte polynomial:

$$T_{\mathcal{M}^*}(x, y) = T_{\mathcal{M}}(y, x).$$

We can also define the Tutte polynomial inductively via contracting/deleting edges [10]; this definition can be easier to work with on graphs.

4.2 Matroid Complexes

To every matroid, $\mathcal{M}$, we associate an abstract simplicial complex (ASC), $\Delta(\mathcal{M})$. The faces of $\Delta(\mathcal{M})$ are the independent sets of $\mathcal{M}$, so the facets are the bases of $\mathcal{M}$. We call an ASC, Δ, a **matroid complex** if $\Delta = \Delta(\mathcal{M})$, for some $\mathcal{M}$.

We have already defined the h-vector of an ASC, Δ,

$$h_\Delta = h(\Delta) = (h_0(\Delta), ..., h_d(\Delta)),$$

$$h_k(\Delta) = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1},$$

$$f_i(\Delta) = |\Delta_i|.$$

So to every matroid we can associate an h-vector, $h_{\Delta(\mathcal{M})}$, as well. In this section, we'll investigate the relationship between the Tutte Polynomial, the h-vector, the number of bases of the matroid, and the size of the critical group.

It is a theorem of Criel Merino's that the Tutte Polynomial of a graph, $T(x, y)$, evaluated at $x = 1$, gives, via the coefficients of the resulting terms, the number of super-stable configurations of the graph of each degree [9]. An easy consequence of this is that $T(1, 1)$ is equal to the number of super-stables, i.e., the size of the critical group of the graph. The Tutte Polynomial is generalized to matroids. For a matroid, $\mathcal{M}$, $T(1, 1)$ is equal to the number of bases of $\mathcal{M}$. So in the graphic case, the size of the critical group is the number of bases. However, this is not generally true for non-graphic matroids. As we demonstrated with Theorem 3.1, $U(2k, 3)$ has critical group of order 2, but it has $\binom{2k}{3}$ bases. We haven't yet found a counterexample for the following conjecture, however:

Conjecture. *The order of the critical group of an oriented matroid divides the sum of its h-vector.*

4.3 Stanley's Conjecture

Definition. *A vector $h = (h_1, ..., h_n) \in \mathbb{Z}^n$ is called a* **pure O-Sequence** *if it is the degree sequence of some* **pure multicomplex.**

A **pure multicomplex** is an order ideal of monomials with a set of maximal (with respect to divisibility) elements of common degree d. An **order ideal** of a poset, P, is a subset, $I \subset P$, with $y < x, x \in I \Rightarrow y \in I$.

Stanley's Conjecture. *The h-vector of a matroid complex is a pure O-Sequence.*

Merino proves Stanley's conjecture for cographic matroids, using the result that the super-stables of a graph form an order ideal. The conjecture has also been proved for several other classes of matroids [19],[20].

Merino's proof also rests of the following relation between h-vectors.

$$\begin{aligned} h_G(t) :&= T_G(1, t) \\ &= T_{\mathcal{M}(G)}(1, t) \\ &= T_{\mathcal{M}^*(G)}(t, 1) \\ &= h_{\Delta(\mathcal{M}^*)}(t). \end{aligned}$$

where $\mathcal{M}(G)$ is the cycle matroid, as defined in 3.1.2. The identity

$$T_{\mathcal{M}(G)}(1, t) = T_{\mathcal{M}^*(G)}(t, 1) = h_{\Delta(\mathcal{M}^*)}(t)$$

is in Merino's Thesis [9]. This relation, $T_{\mathcal{M}}(1, t) = h_{\Delta(\mathcal{M}^*)}(t)$, does *not,* in general, hold for non-graphic matroids; the uniform matroid $U(8, 3)$ provides a counterexample.

Chapter 5

Realizations of Graphic Matroids Using Hypergraphs

In trying to prove that a set of point configurations, $X_1, X_2, ...$ represented the matroids $\mathcal{M}(K_1), \mathcal{M}(K_2), ...$, where K_n is the complete graph on n vertices, I stumbled across what I believe to be a more general way of realizing (at least some) graphic matroids. This method of realization is distinct from the representation given in Proposition 5.1.2 of [29]; it gives a point configuration representing the $\mathcal{M}$, rather than a vector configuration, to name one difference. Briefly, for my method, we consider a graph, G, as a hypergraph, $\mathcal{G}$, and then in the complement of the dual of $\mathcal{G}$, denoted $(\mathcal{G}^*)^c$, we consider hyperedges as hyperplanes and vertices as points, and containment in the hypergraph as geometric containment. It remains to see if every graph can be translated into a set of hyperplanes in this way. First I'll need to go over some hypergraph basics.

5.1 Hypergraph Basics

Hypergraphs are perhaps the most basic combinatorial objects. A hypergraph is a generalization of a graph which allows edges (called **hyperedges**) to contain (or "connect") any number of vertices. Avoiding many technicalities, we define a **hypergraph,** $H = \{V, \mathcal{E}\}$, with $V = \{v_1, ..., v_n\}$ a finite set, and $\mathcal{E} = \{e_1, ..., e_m\}$, where each e_i **contains** a subset of V. The edges need not contain distinct subsets. We can represent a hypergraph, H, via its incidence matrix:

$$A_H = (a_{ij}),$$

where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \in e_j \\ 0 & \text{otherwise.} \end{cases}$$

Conversely, every $n \times m$ matrix, A, with entries in $\{0, 1\}$, determines a hypergraph, $H(A)$. Every hypergraph, H, has a dual, H^*, with the property that $H^{**} = H$. The dual is easily defined via the incidence matrix. Let $\mathcal{H}$ be a hypergraph, and define:

$$\mathcal{H}^* = H(A_{\mathcal{H}}^T).$$

Any graph, G, is trivially a hyperpgraph, $\mathcal{G}$, with the same vertices and edges. We will use $\mathcal{G}$ always to mean the hypergraph corresponding to G. In particular, this is significant when considering duality, since the notions do *not* coincide.

We use H^c to denote the complement of the hypergraph H. This is the hypergraph we get by swapping all the 0's and 1's in the adjacency matrix.

$$A_H^c = 1_{n\times m} - A_H.$$

Where $1_{n\times m}$ is the matrix of 1's.

Notice that dualizing and complementing commute, as a result of the linearity of the matrix transpose map:

$$A_{(H^c)^*} = A_{H^c}^T = (1_{n\times m} - A_H)^T = 1_{m\times n} - A_H^T = 1_{m\times n} - A_{H^*} = A_{(H^*)^c}.$$

5.2 Hypergraphs from Hyperplanes

An affine hyperplane (we'll just call them hyperplanes) is an affine subspace of codimension 1. Hyperplanes can be given by a normal vector, $a = (a_1, ..., a_n)$ and a displacement scalar, b. The hyperplane determined by a and b is then the set of points $x = (x_1, ..., x_n)$ satisfying

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b$$

Hyperplanes generalize planes (in $\mathbb{R}^3$) and lines (in $\mathbb{R}^2$).

Let $F = \{f_1, ..., f_{n+2}\}$, a set of (distinct) hyperplanes of $\mathbb{R}^n$. In this section, we define

$$X = X(F) = \{x \in \mathbb{R}^n \,|\, \exists F_x \subset F \text{ s.t. } \cap F_x = \{x\}\}$$

So X is the set of all points $x \in \mathbb{R}^n$ that can be written (as a singleton set) as the intersection of some of the hyperplanes of F. Now we can define $\mathcal{M}(F) := \mathcal{M}(X)$.

To translate between graphs and point configuration, we define the **incidence hypergraph** of F, using these points as vertices, the hyperplanes of F as edges, and containment as containment.

$$H = H(F) = \{X, \mathcal{E}\}$$

$$\mathcal{E} = \{\mathfrak{f}_1, ...\mathfrak{f}_{n+2}\},$$

where the contents of $\mathfrak{f}_i$ are exactly $X \cap f_i$. This translation corresponds vertices of the graph, G, to hyperplanes of F, and edges of the graph to points of $X(F)$, but in a less-than-inuitive way. A point is contained in a hyperplane iff the corresponding edge does *not* contain the corresponding vertex. Thus, the fact that in a graph every edge contains 2 vertices translates to the requirement that every point of X lies in exactly n of the hyperplanes. So then for any 2 hyperplanes, f_i, f_j, to see if their vertices v_i, v_j are connected in G, we check if their union contains all the points of X :

$$X \subset f_i \cup f_j \text{ iff } \{v_i, v_j\} \notin E(G).$$

Now, we have the following novel conjecture, hopefully soon to be a theorem.

Conjecture 5.1. *Let G be a graph, $\mathcal{G}$ the corresponding hypergraph. Let F be a set of hyperplanes in $\mathbb{R}^n$ such that $H(F) = (\mathcal{G}^*)^c$. Then $\mathcal{M}(G) = \mathcal{M}(F)$.*

A related question is whether for any graph, G, there is a set of hyperplanes, F, such that $H(F) = (\mathcal{G}^*)^c$. Currently I've had problems in generating such an F for the **house graph**, which is C_5 (the cycle on 5 vertices) plus some additional edge. Also, every graph I have constructed so far has been of a particular type: Given the complete graph, K_n, and a set of disjoint subsets, $A = \{A_1, ..A_k\}$, of the vertices, we define $K_n \setminus A$ by removing all edges $e = \{u, v\}$, such that there exists an A_i containing both u and v. All of my examples so far are of the form $K_n \setminus A$, for some A and some $n \leq 5$. These suggest possible limitations to the answer to this question.

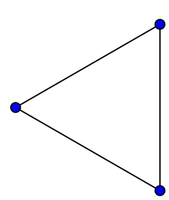
5.3 Examples

Here we give some examples of the following translation. I've found examples for the following:

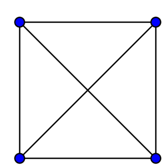
1. K_3
2. K_4
3. $K_4 \setminus \{v_1, v_2\}$
4. $K_4 \setminus \{\{v_1, v_3\}, \{v_2, v_4\}\}$
5. K_5
6. $K_5 \setminus \{v_1, v_2\}$
7. $K_5 \setminus \{v_3, v_4, v_5\}$
8. $K_5 \setminus \{\{v_1, v_2\}, \{v_3, v_4, v_5\}\}$

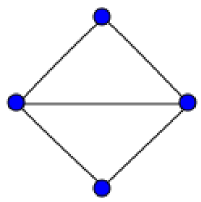
Next is a list of figures for the corresponding graphs:

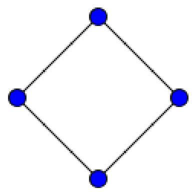
1.

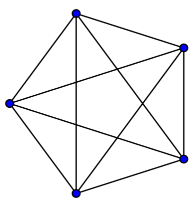


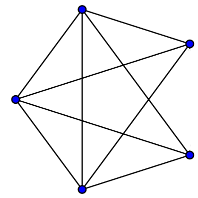
2.

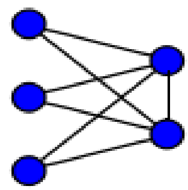


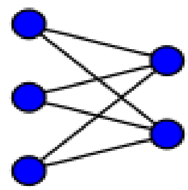
3. 

4. 

5. 

6. 

7. 

8. 

Next is a list of equations of some corresponding hyperplane arrangements, in order:

1. $x = 0$, $x = 1$, $x = 2$.

2. $x = 0$, $y = 0$, $x - y = 1$, $x + y = 4$.

3. $y = 1$, $y = 3$, $x - y = 0$, $x + y = 4$.

4. $x = 0$, $x = 1$, $y = 0$, $y = 1$.

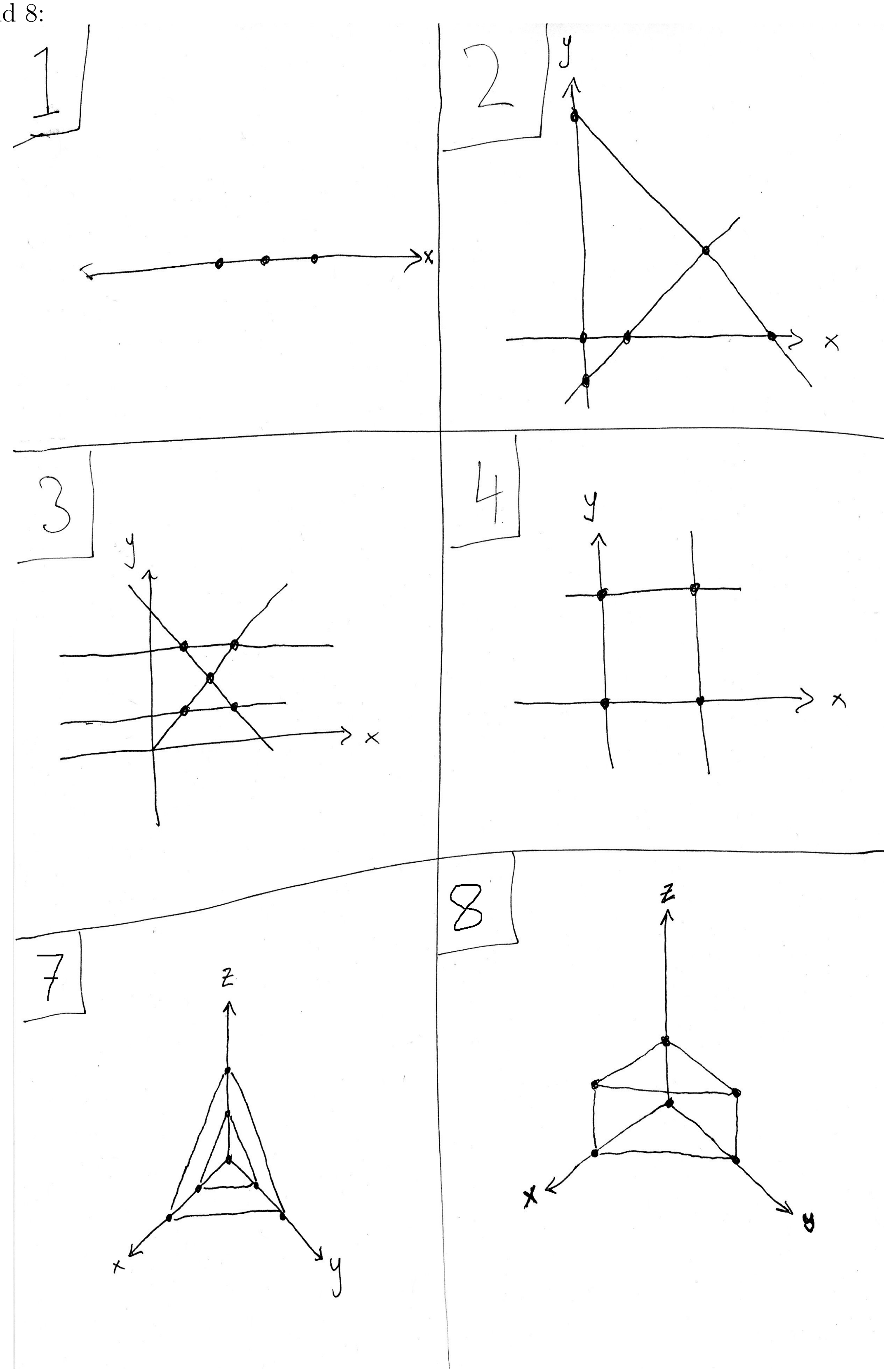
5. $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$, $3x + 2y + z = -6$.

6. $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$, $2x + y + z = 4$.

7. $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$, $x + y + z = 2$.

8. $x = 0$, $y = 0$, $z = 0$, $z = 1$, $x + y = 2$.

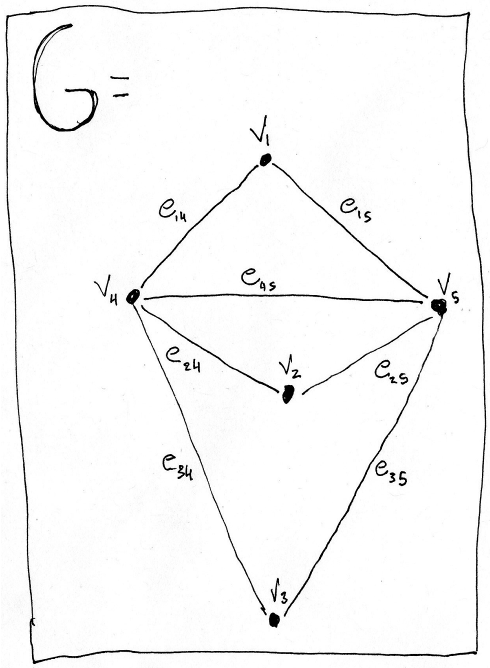
Here is a list of drawings of I made of the hyperplane arrangements for 1,2,3,4,7, and 8:



Finally, here we see a labelled example for the graph $G = K_5 \setminus \{v_3, v_4, v_5\}$, showing the correspondences. Define $F = \{h_1, h_2, h_3, h_4, h_5\}$ with:

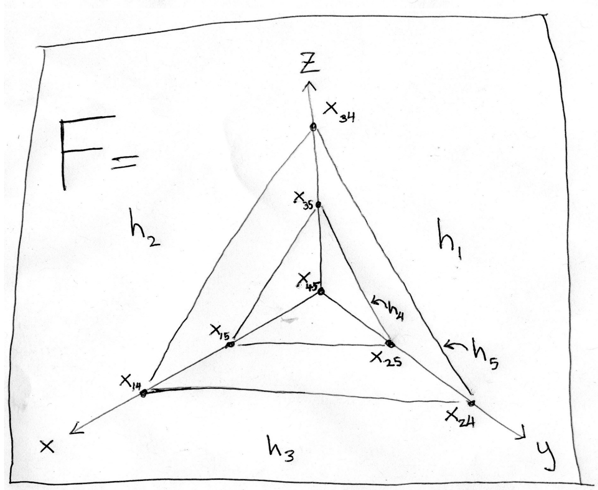
* $h_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$,
* $h_2 = \{(x, y, z) \in \mathbb{R}^3 \mid y = 0\}$,
* $h_3 = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$,
* $h_4 = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1\}$,
* $h_5 = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 2\}$,

Then we have the following:



$$A_{\mathcal{G}} = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} \begin{array}{ccccccc} e_{14} & e_{15} & e_{24} & e_{25} & e_{34} & e_{35} & e_{45} \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{array}$$

$$A_{(\mathcal{G}^*)^c} = \begin{array}{c} \\ V_{14} \\ V_{15} \\ V_{24} \\ V_{25} \\ V_{34} \\ V_{35} \\ V_{45} \end{array} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{array}$$



References

1 A. Björner, M. Las Vergnas, B. Sturmfels, N. White, G. Ziegler. *Oriented Matroids.* Cambridge University Press, 1993.

2 A. Duval, C. Klivans, J. Martin. “Critical Groups of Simplicial Complexes”.

3 A. Duval, C. Klivans, J. Martin. “The Critical Group of a Simplicial Complex”. *powerpoint slides from a talk given at Texas State University on April 24, 2010.*

4 A. Duval, C. Klivans, J. Martin. “Simplicial Matrix-Tree Theorems”.

5 Bernd Sturmfels. *Gröbner Bases and Convex Polytopes.* American Mathematical Society, 1996.

6 C. Godsil and G. Royle. *Algebraic Graph Theory.* Springer-Verlag, 2001.

7 C. Merino, S. Noble, M. Ramirez-Ibañez, R. Villarroel. “On the Structure of the h-Vector of a Paving Matroid”.

8 Criel Merino. “The Chip Firing Game and Matroid Complexes”.

9 Criel Merino. *Matroids, the Tutte Polynomial, and the Chip-Firing Game.*

10 David Perkinson. *Course notes from Topics in Analysis: Sandpiles (Math 412 Reed College Spring 2010)*

11 David Perkinson. *personal correspondence.*

12 David Perkinson. “Duality for Sandpiles”. June 28, 2010.

13 Federico Ardilla. “Subdominant matroid ultrametrics”.

14 Günter Ziegler. *Lectures on Polytopes.* Springer-Verlag, 1995.

15 Hatcher, Allen. *Algebraic Topology.* Cambridge University Press, 2002.

16 J. Richter-Gebert and G. Ziegler. “Oriented Matroids”.

17 L. Finschi and F. Komei. “Complete Combinatorial Generation of Small Point Configurations and Hyperplane Arrangements”.

18 Martin Rubey. *Counting Spanning Trees.*

19 Suho Oh. "Permutohedra, h-vector of cotransversal matroids and pure O-sequences".

20 Suho Oh. "Research Statement".

21 W. Cook, V. Reiner, and D. Stanton. "Combinatorial Laplacians of Matroid Complexes".

22 W. Hochstättler and R. Nickel. "The Flow Lattice of Oriented Matroids".

23 A. Gyárfás. *Advanced Combinatorics Handouts*. Budapest Semesters in Mathematics: 2009 Fall.

24 "Homepage of Oriented Matroids". *http://www.om.math.ethz.ch/*.

25 A. Holroyd, L. Levine, K. Mészarós, Y. Peres, J. Propp, and D. Wilson. "Chip-Firing and Rotor-Routing on Directed Graphs". arXiv: 0801.3306v3.

26 L. Batten and A. Beutelspacher. *The Theory of Finite Linear Spaces*. Cabridge University Press: 1993.

27 B. Grünbaum. *Configurations of Points and Lines*. American Mathematical Society: 2009.

28 P. McMahan. *Wreath-Product Polytopes*. 2003.

29 J. Oxley. *Matroid Theory*. Oxford University Press: 1992.

30 "Gallery of Named Graphs" *http://en.wikipedia.org/wiki/Gallery_of_named_graphs*.