# Divisor Theory of Simplicial Complexes

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# Abstract

The theory of divisors on graphs is a subject that has been discovered independently through a variety of mathematical models. It closely resembles the theory of divisors on algebraic curves in the field of algebraic geometry and can be thought of as being a discrete version of algebraic curve divisors; many results from the algebraic geometry theory of divisors have an analog in the graph-theoretic version. A higher-dimensional generalization of divisor theory also exists, in which graphs are replaced by simplicial complexes, though not as much is known about this generalization. This thesis will begin with an overview of the theory of divisors on graphs, including definitions and proofs of a variety of important results. This thesis will then introduce the theory of divisors on simplicial complexes and prove a few new results for this theory, analogous to the results on graphs.

# Introduction

The dollar game is a simply stated but structurally complex game played on the vertices and edges of a graph. It is one of many equivalent or related mathematical models on a graph that have been discovered independently in various fields such as statistical physics, probability theory, and graph theory. The dollar game and its equivalent formulations are particularly interesting because they can be used to develop a theory of divisors on graphs that closely mirrors the theory of divisors on algebraic curves in the field of algebraic geometry. The parallel between algebraic geometry and graph theory goes so far as to include a graph-theoretic analog, proven by Baker and Norine [1] in 2006, of the cornerstone Riemann-Roch theorem of algebraic geometry. In 2011, [3] introduced a generalization of the theory of divisors on graphs to a theory of divisors on simplicial complexes. Much less is known about the theory of divisors on simplicial complexes than about the theory of divisors on graphs and the aim of this thesis will be to reduce that gap.

Chapter 1 of this thesis will introduce the theory of divisors on graphs through the dollar game model and then provide an overview of definitions and a variety of results about divisors on graphs. Chapter 2 will introduce Duval, Klivans, and Martin's [3] generalization to a theory of divisors on simplicial complexes and prove a variety of new results in this theory.

# Chapter 1

# Divisors on Graphs

## 1.1 The Dollar Game

To introduce the subject of divisors, we will begin with a simple game played on a graph, called the dollar game. To describe it, we must first define what a graph is:

**Definition 1.1.** A **graph** $G$ consists of a set $V$ of vertices and a set $E$ of edges, where an edge is a pair of distinct vertices. A graph is **connected** if for every pair of vertices $v, v'$ there is a sequence of vertices starting with $v$ and ending with $v'$ such that any two adjacent vertices in the sequence share an edge.

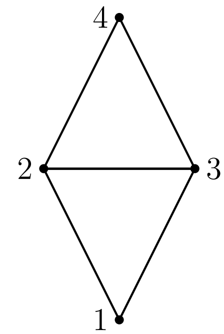
This definition precludes the possibility of multiple edges between two vertices, and an edge from a vertex to itself, so this is more accurately an undirected graph without multiple edges or loops. For the rest of this paper we will take *graph* to mean a connected, undirected graph without multiple edges or loops.

The dollar game on a graph $G$ then proceeds as follows. Initially, every vertex of $G$ has some number of dollars, possibly negative. We can make the following moves:

1. A single vertex of $G$ can simultaneously receive one dollar from each neighboring vertex (vertex it shares an edge with). This is called a borrowing move at a vertex $v$.
2. A single vertex of $G$ can simultaneously give one dollar to each neighboring vertex. This is called a lending move at a vertex $v$.

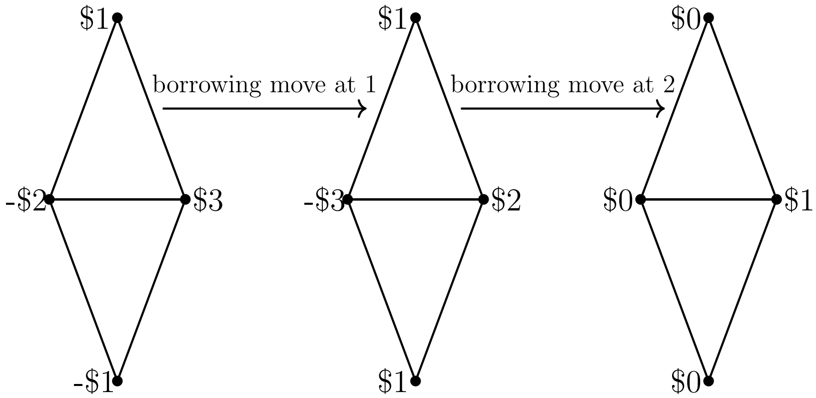
The goal of the game is to end with every vertex having a nonnegative amount of dollars.

**Example 1.1.** As an example, lets consider the dollar game played on the graph to the right, which we will call the diamond graph. It has vertices $V = \{1, 2, 3, 4\}$ and edges $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$.




Suppose we start with $-\$1$ on vertex 1, $-\$2$ on vertex 2, \$3 on vertex 3, and \$1 on vertex 4.

Can this game be won? It turns out the answer is yes; If we borrow at vertex 1 and then borrow at vertex 2, every vertex then has a nonnegative amount of dollars:




The central question the first chapter of this paper will be concerned with is, in the language of the dollar game: For which initial configurations of dollars on the vertices of a graph is the dollar game able to be won?

To answer this question, we will first begin by formalizing the dollar game with the language of divisors.

## 1.2 Definitions

**Definition 1.2.** Let $G = (V, E)$ be a graph. A **divisor** of $G$ is a formal sum of the vertices of $G$ with integer coefficients, i.e., anything of the form

$$D = \sum_{v \in V} D(v)v$$

where each $D(v)$ is an integer. The set of all divisors of $G$ is thus equivalent to $\mathbb{Z}V$, the free abelian group on the vertices of $G$.

If we fix an ordering of the vertices, $V = \{v_1, v_2, \ldots, v_n\}$, we can use it to construct an isomorphism between the set of all divisors of $G$ and $\mathbb{Z}^n$, where $\sum_{v \in V} a_v v \mapsto$

$(a_{v_1}, a_{v_2}, \ldots, a_{v_n})$. For ease of notation, we will often refer to a divisor by its image under this isomorphism, e.g., $2v_1 + 3v_3 = (2, 0, 3)$ in a graph with 3 vertices.

A divisor can be thought of as a formalization of assigning some number of dollars to each vertex of a graph. The coefficient $D(v)$ records the number of dollars at vertex $v$ for a divisor $D$.

**Definition 1.3.** Let $G = (V, E)$ be a graph. A divisor $D$ of $G$ is **effective** if $D(v) \geq 0$ for every vertex $v \in V$.

Effective divisors can be though of as winning states of the dollar game; every vertex has a nonnegative amount of dollars.

**Definition 1.4.** Let $G$ be a graph and $D$ a divisor of $G$. The **degree** of $D$ is the sum of $D(v)$ over all vertices:

$$\deg D = \sum_{v \in V} D(v).$$

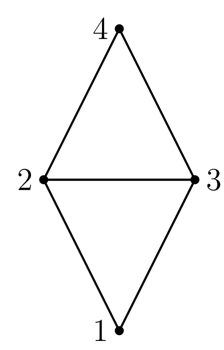
The degree of a divisor is analogous to the total amount of wealth in an instance of the dollar game.

**Definition 1.5.** Let $G = (V, E)$ be a graph with $|V| = n$. The **Laplacian** $L$ of $G$ is a linear function from $\mathbb{Z}^n \to \mathbb{Z}^n$ given by multiplication by the matrix also denoted $L$ where:

$$L_{ij} = \begin{cases} \deg(v_i) & i = j \\ -1 & i \neq j,\ v_i v_j \in E \\ 0 & i \neq j,\ v_i v_j \notin E \end{cases}$$

**Example 1.2.** For the diamond graph, the Laplacian matrix is as follows:

$$\begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$




**Definition 1.6.** Let $G = (V, E)$ be a graph. A **principal divisor** is a divisor in the image of the Laplacian of $G$.

**Definition 1.7.** Let $G = (V, E)$ be a graph, and $D_1, D_2$ divisors on $G$. Then $D_1$ is **linearly equivalent** to $D_2$ if $D_1 - D_2$ is a principal divisor. When $D_1$ is linearly equivalent to $D_2$ we write $D_1 \sim D_2$.

The definition of linear equivalence captures the idea of being able to transform one divisor to another through borrowing and lending moves. Notice that the $i^{th}$ column of the Laplacian encodes a borrowing move at vertex $v_i$. Therefore, a principal divisor can be thought of as possible changes that could be made to a divisor after some number of borrowing and lending moves, and linearly equivalent divisors are those that can be changed into each other through borrowing and lending moves. To highlight this correspondence, we will call $L(e_i)$, where $e_i$ is the $i^{th}$ standard basis vector, a borrowing move at $v_i$, and $L(-e_i)$ a lending move at $v_i$. In this way, the input to the Laplacian can be thought of as describing a script for a list of moves to make in the dollar game; the $i^{th}$ component determines how many times to fire at vertex $v_i$. We will call an input of the Laplacian a **firing script** for $G$.

**Example 1.3.** Reinterpreting our dollar game example in this context, we started with $-1$ dollars on vertex 1, $-2$ dollars on vertex 2, 3 dollars on vertex 3, and 1 dollar on vertex 4, which corresponds to the divisor $(-1, -2, 3, 1)$.

We then had vertices 1 and 2 each perform a borrowing move, corresponding to adding $L(1, 1, 0, 0) = (1, 2, -2, -1)$ to our divisor. This resulted in the effective divisor $(0, 0, 1, 0)$, demonstrating that our starting divisor was winnable.

**Proposition 1.1.** *The degree of any principal divisor is 0.*

*Proof.* This is clear through the dollar game analogy, a borrowing move does not change the total amount of dollars present, so no sum of borrowing moves will. □

**Corollary 1.1.** If $D \sim D'$, then deg $D = $ deg $D'$.

*Proof.* Since $D \sim D'$, it follows that $D = D' + P$ for some principal divisor $P$. Then deg $D = $ deg $D' + $ deg $P = $ deg $D'$, since deg is easily seen to be a linear function on divisors. □

**Definition 1.8.** Let $G = (V, E)$ be a graph. A divisor on $D$ on $G$ is called **winnable** if there exists an effective divisor $D'$ such that $D \sim D'$. A firing script $\sigma$ that realizes this equivalence, i.e., $D + L(\sigma) = D'$, is called a **winning firing script**.

This final definition captures the idea of winning the dollar game, and when a divisor is winnable will be our main focus from now on.

## 1.3 The Greedy Algorithm

We want to know how to determine when a divisor is winnable. One way to do this is an algorithmic method called the Greedy Algorithm. It works as follows:

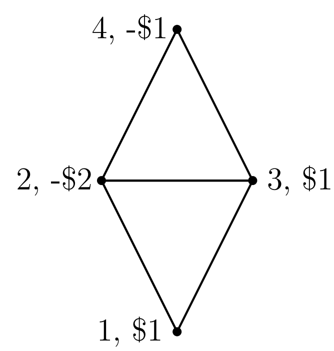
**Theorem 1.1.** *Let $G = (V, E)$ be a graph and $D$ a divisor. The following algorithm computes whether $D$ is winnable:*

1. *If $D(v) \geq 0$ for every vertex $v$, then $D$ is effective and therefore winnable.*
2. *Otherwise, choose a vertex $v$ such that $D(v) < 0$ such that not all vertices other than $v$ have previously been chosen. If this is impossible, $D$ is not winnable.*
3. *Perform a borrowing move at $v$, i.e., add the column of $L$ corresponding to $v$ to $D$ to create a divisor $D'$.*
4. *Repeat this algorithm with $D'$ instead of $D$, retaining the information about which vertices have been chosen for a borrowing move.*

For example, if we were to run the greedy algorithm on our dollar game example, it would produce the same sequence of moves we used and demonstrate winnability. An example where the greedy algorithm gives a negative answer is as follows:

**Example 1.4.** Consider the divisor $(1, -2, 1, -1)$ on the diamond graph. The greedy algorithm could proceed as follows:

1. Borrow at vertex 2, resulting in the divisor $(0, 1, 0, -2)$.
2. Borrow at vertex 4, resulting in the divisor $(0, 0, -1, 0)$.
3. Borrow at vertex 3, resulting in the divisor $(-1, -1, 2, -1)$.
4. Borrow at vertex 2, resulting in the divisor $(-2, 2, 1, -2)$.
5. Borrow at vertex 4, resulting in the divisor $(-2, 1, 0, 0)$.




Now we have borrowed at every vertex except 1 already, so the greedy algorithm will determine this divisor to be unwinnable.

To prove that this algorithm works, we use the following proposition:

**Proposition 1.2.** *Let $G$ be a graph and $L$ its Laplacian. Then the kernel of $L$ is the span of $\sigma = (1, 1, \ldots, 1)$. Equivalently, the only way to leave a divisor unchanged is to perform the same number of borrowing or same number of lending moves at each of the vertices.*

*Proof.* Consider a divisor $D$ and a single vertex $v$ of $G$. After borrowing at every vertex, we will have increased $D(v)$ by 1 for each edge containing $v$ when we borrow at $v$, and decreased $D(v)$ by 1 for each edge $vw$ when we borrow at $w$, for net 0 effect.

Therefore $L(1, 1, \ldots, 1) = 0$. To see that nothing else lies in the kernel, consider $\sigma \in \ker L$. Let $v$ be a vertex for which $\sigma(v)$ is maximal, and let $k$ be its maximum value. Then

$$0 = L(\sigma)(v) = k \deg(v) - \sum_{vw \in E} \sigma(w).$$

Now each $\sigma(w) \leq k$, so the only way to achieve this equality is for $\sigma(w) = k$ for each $w$ neighboring $v$. Since $G$ is connected, iterating this argument for all $w$ neighboring $v$ gives that $\sigma = (k, k, \ldots, k)$. □

*Proof of Theorem 1.1.* Firstly, it is clear that if the algorithm finds $D$ to be winnable, it does so by producing a winning firing script, so $D$ is winnable. It remains to show that the algorithm always terminates and determines every winnable divisor to be winnable.

If $D$ is winnable, then there exists some winning firing script $\sigma$. By adding the correct multiple of $(1, 1, \ldots, 1)$ we can construct $\sigma' = \sigma + n(1, 1, \ldots, 1)$ such that $\sigma'$ is nonnegative in each component and 0 in some component. By Proposition 1.2, $D + L(\sigma') = D + L(\sigma)$ is effective, so $\sigma'$ is a winning firing script for $D$. Now if $\sigma'_i = 0$, then $D(v_i)$ cannot be negative, as the only way to increase $D(v_i)$ using only borrowing moves is to borrow at $v_i$. Therefore, after we borrow at vertex $v_i$ as in the greedy algorithm, to transform $D$ into $D'$, $\sigma' - e_i$ is now a winning firing script for $D'$, and retains the property that each component is nonnegative. Eventually we must reach the point where the winning firing script is 0, and we thus have an effective divisor.

If $D$ is not winnable, consider the set of all divisors $S$ that the algorithm iterates through. For each $D' \in S$, and each vertex $v \in V$, we have

$$D'(v) \leq \max(D(v), \deg(v) - 1).$$

This is because the only way to increase $D'(v)$ is to borrow at $v$, which only occurs when the divisor is negative at $v$. Additionally, since the degree of all divisors in $S$ is constant, there is a minimum value of $D'(v)$ at each $v$. Therefore $S$ is a finite set. Consequently, if the algorithm does not terminate, it repeats some divisor twice. But by Proposition 1.2, this can only happen by performing the same number of borrowing moves at each vertex, which contradicts the behavior of the algorithm. □

## 1.4 Degree, Rank, and Riemann-Roch

While the greedy algorithm will determine if a divisor is winnable, it is useful to have non-algorithmic methods of determining whether a divisor is winnable. One such way is to look at the degree of the divisor. For example, the degree of the divisor can sometimes immediately show that a divisor is not winnable:

**Proposition 1.3.** *Let $G$ be a graph and $D$ a divisor on $G$ with $\deg(D) < 0$. Then $D$ is not winnable.*

*Proof.* Let $D'$ be an effective divisor. Then $\deg(D') = \sum_{v \in V} D'(v) \geq 0$, so $\deg(D) \neq \deg(D')$ and therefore $D$ is not linearly equivalent to $D'$. □

More surprising is the fact that the degree of the divisor can also sometimes immediately show that a divisor is winnable. Intuitively, the idea is that if the total amount of dollars in the graph is large enough, it should always be possible to win the dollar game. Formally, this is stated:

**Theorem 1.2.** *Let $G$ be a graph. Then there exists an integer $w$ such that every divisor $D$ with $\deg(D) \geq w$ is winnable.*

This is particular case of Theorem 2.3, and the proof will be deferred until then. While I will not prove it here, $w$ is actually quite easy to compute for a given graph $G = (V, E)$. It turns out that $w = |E| - |V| + 1$, a source for this fact can be found in [2]. In particular, if $G$ is a tree, $w = 0$, meaning that for a tree, the degree of a divisor completely determines if it is winnable.

This theorem motivates the following definition, which can be thought of as a sort of measure of how winnable a divisor $D$ is:

**Definition 1.9.** Let $G$ be a graph, $D$ a divisor on $G$. The **rank** of $D$, $r(D)$ is either $-1$ if $D$ is unwinnable, or if $D$ is winnable, it is the largest integer $r$ such that for any effective divisor $\lambda$ of degree $r$, $D - \lambda$ is winnable.

Theorem 1.2 tells us that there exist divisors with arbitrarily large rank. This definition serves to reinforce the analogy between divisors on graphs and divisors of algebraic curves, as rank is an important part of the algebraic geometry theory of divisors. The rank of a divisor is a central part of one of the cornerstone theorems of algebraic geometry, the Riemann-Roch theorem. Somewhat miraculously, there exists a combinatorial analog to this theorem, proven by Baker and Norine in [1]:

**Theorem 1.3** (Riemann-Roch for Graphs)**.** *Let $G$ be a graph, $D$ a divisor on $G$, and let $K$ denote the divisor with $K(v) = deg(v) - 2$ for each vertex $v$ in $G$. Then*

$$r(D) - r(K - D) = deg(D) - w + 1$$

*where $w = |E| - |V| + 1$ as in Theorem 1.2.*

*Proof.* See Baker and Norine [1] for proof. □

# Chapter 2

# Divisors on Simplicial Complexes

The theory of divisors on graphs can be though of as a discrete analog of the theory of divisors on algebraic curves, which has a generalization to divisors of higher-dimensional algebraic varieties. It is natural to then ask if the theory of divisors on graphs can be generalized to higher dimensions as well. Duval, Klivans, and Martin [3] introduced such a generalization with their theory of divisors on simplicial complexes. This chapter will introduce the theory of divisors on simplicial complexes and prove some new results about simplicial divisors analogous to the results from Chapter 1.

## 2.1 Definitions

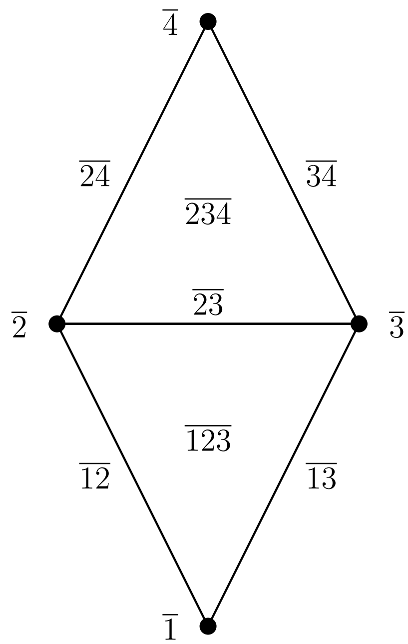
**Definition 2.1.** A **simplicial complex** $C$ is a finite collection of finite sets that is closed under taking subsets, i.e., if $B \in C$ and $A \subset B$, then $A \in C$.

The elements of $C$ are called the **faces** of the simplicial complex. The **dimension** of a face $f$ is $|f| - 1$ , and the 0-dimensional faces of a simplicial complex are its **vertices**. Any face of $C$ that is not a subset of any other face of $C$ is called a **facet** of $C$, and we will often refer to a complex simply by its facets, as the rest of the faces are determined by them.

**Example 2.1.** We can turn the diamond graph into the diamond simplicial complex by adding two 2-dimensional faces $\{1,2,3\}$ and $\{2,3,4\}$.
The faces of this simplicial complex are

| | |
|---|---|
| $(-1)$-d : | $\emptyset$ |
| 0-d : | $\{1\},\{2\},\{3\},\{4\}$ |
| 1-d : | $\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{3,4\}$ |
| 2-d : | $\{1,2,3\},\{2,3,4\}$. |

The facets are $\{1,2,3\},\{2,3,4\}$.




For our purposes, we also require a total order relation on the vertices of $C$. For ease of notation, we will refer to a vertex by its position in the total order with a bar, e.g., $\overline{3}$, and a face by the vertices it contains, in order, e.g., $\overline{135}$.

Note that if a simplicial complex has only 0-dimensional and 1-dimensional faces, it contains the same information as a graph. In this way, simplicial complexes can be thought of as higher-dimensional generalizations of graphs. This idea motivates the following definition:

**Definition 2.2.** The **1-skeleton** or **underlying graph** of a simplicial complex $C$ is the graph $G = (V, E)$ where $V$ is the set of vertices of $C$ and $E$ is the set of 1-dimensional faces of $C$.

**Example 2.2.** The diamond simplicial complex from Example 2.1 has the diamond graph as its underlying graph.

**Definition 2.3.** An $i$-dimensional **divisor** $D$ of a simplicial complex $C$ is an element of the free abelian group on the $i$-dimensional faces of $C$, i.e. ,

$$D = \sum_{f \text{ an } i\text{-dimensional face of } C} D(f)f.$$

The set of all $i$-dimensional divisors of a simplicial complex $C$ is denoted by $C_i$.

As before, for ease of notation we will identify $C_i$ with $\mathbb{Z}^n$, where $n$ is the number of $i$-dimensional faces of $C$. We will write $D = (D(f_1), \ldots D(f_n))$, where the $i$-dimensional faces are ordered in lexicographic order with respect to the vertex ordering.

This definition of a divisor on a simplicial complex generalizes the definition for a divisor on a graph, since a 0-dimensional divisor of a simplicial complex is the same as a divisor on its underlying graph.

We also want a generalization of linear equivalence to higher-dimensional divisors, which the next few definitions will provide.

**Definition 2.4.** The $i^{th}$ **boundary map** of a simplicial complex $C$, denoted by $\partial_i : C_i \to C_{i-1}$, is defined on faces as follows:

$$\partial_i(\overline{v_0 v_1 \cdots v_i}) = \sum_{k=0}^{i} (-1)^k \overline{v_0 v_1 \cdots \widehat{v_k} \cdots v_i}$$

and extended linearly to $C_i$.

For example, in the diamond simplicial complex, $\partial_2(\overline{234}) = \overline{34} - \overline{24} + \overline{23}$.

The $C_i$ and $\partial_i$ form a sequence called the chain complex of $C$:

$$C_i \xrightarrow{\partial_i} C_{i-1} \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \longrightarrow 0$$

It is called a chain complex because $\partial_{i-1}\partial_i = 0$:

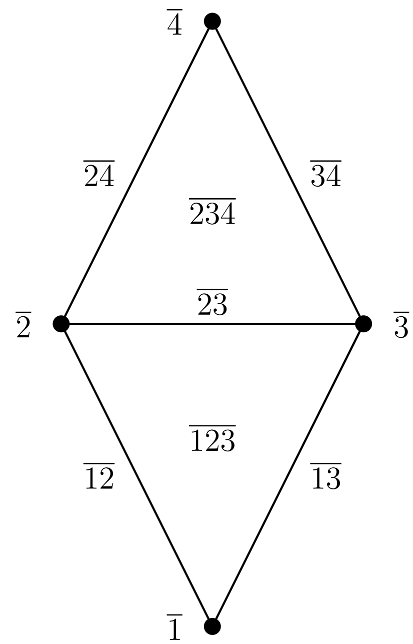
$$\partial_{i-1}(\partial_i(\overline{v_0 v_1 \cdots v_i})) = \sum_{j,k \in \{0,\ldots,i\},\ j \neq k} \begin{cases} (-1)^k(-1)^j \overline{v_0 v_1 \cdots \widehat{v_j} \cdots \widehat{v_k} \cdots v_i} & \text{if } j < k \\ (-1)^k(-1)^{j-1} \overline{v_0 v_1 \cdots \widehat{v_k} \cdots \widehat{v_j} \cdots v_i} & \text{if } j > k \end{cases}.$$

This is zero since the $j = a, k = b$ term is the negative of the $j = b, k = a$ term.

**Definition 2.5.** Let $C$ be a simplicial complex. The $i^{th}$ **Laplacian** $L_i : C_i \to C_i$ of $C$ is the map $\partial_{i+1}\partial_{i+1}^t$.

**Example 2.3.** Consider the diamond simplicial complex again. The mapping $\partial_2$ is given by the following matrix, with bases of faces in lexicographical order:

$$\partial_2 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}$$




So $L_1 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 \\ 1 & -1 & 2 & -1 & 1 \\ 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

Thus, for example, $L_1(\overline{23}) = \overline{12} - \overline{13} + 2 \cdot \overline{23} - \overline{24} + \overline{34}$.

It is not hard to verify that the $0^{th}$ Laplacian $L_0$ of a simplicial complex $C$ coincides with the Laplacian of the underlying graph of $C$, so this is a reasonable generalization of the Laplacian to higher dimensions.

With higher-dimensional Laplacian operators defined, we can define linear equivalence and winnability analogously to the 0-dimensional case:

**Definition 2.6.** Let $C$ be a simplicial complex, and $D$ an $i$-dimensional divisor of $C$. Then $D$ is **effective** if $D(f) \geq 0$ for every $i$-dimensional face $f$ of $C$.

**Definition 2.7.** Let $C$ be a simplicial complex, and $D$ an $i$-dimensional divisor of $C$. Then $D$ is a **principal divisor** if $D = L_i(\sigma)$ for some $\sigma \in C_i$.

**Definition 2.8.** Let $C$ be a simplicial complex, and $D, D'$ be an $i$-dimensional divisors of $C$. Then $D$ and $D'$ are **linearly equivalent** if $D - D'$ is a principal divisor. As before, we denote linear equivalence by $D \sim D'$.

**Definition 2.9.** Let $C$ be a simplicial complex, and $D$ an $i$-dimensional divisor of $C$. Then $D$ is **winnable** if there exists an $i$-dimensional divisor $D'$ on $C$ such that $D'$ is effective and $D \sim D'$.

The goal of the following sections will be to reexamine the question of when a divisor is winnable, for higher order divisors.

## 2.2 The Positive Kernel

This section discusses a set that will be necessary to understand for the following sections: the positive kernel of a simplicial complex.

**Definition 2.10.** Let $C$ be a simplicial complex. The $i^{th}$ **positive kernel** is the set

$$\{(x_1, \ldots, x_n) \in \mathbb{Z}^n \mid L_i(x_1, \ldots, x_n) = 0, x_k \geq 0 \text{ for } k = 1, \ldots, n\}.$$

The significance of this set will become apparent later, but first we will prove some results about the positive kernel. To do so, we must first discuss rational convex cones.

**Definition 2.11.** A **convex cone** is a set $Q \subset \mathbb{R}^n$ that is closed under linear combinations with nonnegative coefficients. That is, if $x, y \in Q$ and $\alpha, \beta \geq 0$, then $\alpha x + \beta y \in Q$

**Definition 2.12.** A set of points $X = \{x_1, \ldots, x_k\}$ **finitely generates** a cone $Q$ if

$$Q = \{\alpha_1 x_1 + \cdots + \alpha_n x_n \mid \alpha_i \geq 0\}.$$

A cone is called **rational** and **polyhedral** if there exists a set of integer valued points that finitely generates it.

Though our cones live in $\mathbb{R}^n$, we will be concerned with the set of integer valued points in a cone. Relevant to this set is the concept of a Hilbert basis of a cone:

**Definition 2.13.** Let $Q$ be a rational polyhedral cone. A finite set $\{x_1, \ldots, x_k\} \subset \mathbb{Z}^n$ is a **Hilbert basis** for $Q$ if every integer-valued point in $Q$ can be written as a linear combination of $x_1, \ldots, x_k$ with nonnegative integer coefficients.

**Theorem 2.1** (Uniqueness and Existence of Hilbert bases)**.** *Let $Q$ be a rational polyhedral cone that is* ***pointed****, meaning that $Q \setminus \{0\}$ is contained within some open half-space of $\mathbb{R}^n$. Then there exists a unique minimal Hilbert basis of $Q$ (minimal relative to taking subsets).*

*Proof.* Existence: Let $\{x_1, \ldots, x_k\}$ be a finite set of integral generators of $Q$. Then let $y_1, \ldots, y_t$ be the set of all integral points in

$$\{\lambda_1 x_1 + \cdots + \lambda_k x_k \mid 0 \leq \lambda_i \leq 1, i = 1, \ldots, k\}.$$

We know that there are finitely many $y_i$ because the above set is bounded. Then $\{y_1, \ldots, y_t\}$ forms a Hilbert basis for $Q$. To see this, consider an arbitrary integral point

$$x = \alpha_1 x_1 + \cdots + \alpha_k x_k \in Q$$

with $\alpha_i \geq 0$. Then

$$x = \lfloor \alpha_1 \rfloor x_1 + \cdots + \lfloor \alpha_k \rfloor x_k + (\alpha_1 - \lfloor \alpha_1 \rfloor)x_1 + \cdots + (\alpha_k - \lfloor \alpha_k \rfloor)x_k.$$

Now each of $x_1, \ldots, x_k$ and $(\alpha_1 - \lfloor \alpha_1 \rfloor)x_1 + \cdots + (\alpha_k - \lfloor \alpha_k \rfloor)x_k$ is in $\{y_1, \ldots, y_t\}$, so $x$ is a nonnegative linear combination of $y_i$.

Uniqueness: Consider the set

$$H = \{x \in Q \mid x \neq 0, x \in \mathbb{Z}^n, x \text{ is not a sum of two nonzero integral points in } Q\}.$$

Clearly this set is contained in any Hilbert basis of $Q$, so if it itself is a Hilbert basis, then it is the unique minimal Hilbert basis. Since $Q \setminus \{0\}$ is contained in some open half-space of $\mathbb{R}^n$, there must be a vector $b$ such that $b \cdot x > 0$ for all $x \in Q, x \neq 0$. Now suppose there is an integral point $a \in Q$ that is not a nonnegative linear combination of points in $H$, and suppose $b \cdot a$ is minimal over all such points. Since $a \notin H$, $a = a_1 + a_2$ for some $a_1, a_2 \in Q$. But then $b \cdot a_1 = b \cdot a - b \cdot a_2 < b \cdot a$ so $a_1$ is a nonnegative linear combination of points in $H$ by minimality of $a$. Similarly $a_2$ is a nonnegative linear combination of points in $H$, so $a$ must also be. Therefore $H$ is the unique minimal Hilbert basis for $Q$. □

The relevance of the theory of rational cones to our problem lies in the fact that the positive kernel is in fact the set of integral points of a pointed polyhedral rational cone:

**Proposition 2.1.** *Let $C$ be a simpicial complex and $L_i$ its $i^{th}$ Laplacian. Let $L_i^{\mathbb{R}} : \mathbb{R}^n \to \mathbb{R}^n$ be the linear extension of $L_i$ to $\mathbb{R}^n$. Then $Q = \ker L_i^{\mathbb{R}} \cap \mathcal{O}^+$ is a pointed polyhedral rational cone, where $\mathcal{O}^+$ denotes the* ***positive orthant:*** *$\mathcal{O}^+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, i = 1, \ldots, n\}$.*

*Furthermore, the set of integral points of $Q$ is exactly the $i^{th}$ positive kernel of $C$.*

*Proof.* It is clear by the definition of $Q$ and the positive kernel that the positive kernel is the set of integral points of $Q$. Now $\ker L_i$ is a subspace of $\mathbb{R}^n$ which has integral valued generators, so it is a rational polyhedral cone. The set $\mathcal{O}^+$ is generated as a cone by the standard basis vectors, so it is also a rational polyhedral cone. The intersection of two rational polyhedral cones is again a rational polyhedral cone, so $Q$ is a rational polyhedral cone. Additionally,

$$Q \setminus \{0\} \subset \mathcal{O}^+ \setminus \{0\} \subset \{x \in \mathbb{R}^n \mid x \cdot (1, 1, 1, \ldots, 1) > 0\},$$

so $Q$ is pointed. □

The theory of rational cones therefore allows us to discuss the unique minimal Hilbert basis of the $i^{th}$ positive kernel of $C$, which we will refer to as the $i^{th}$ Hilbert basis of $C$. The remainder of the section will be dedicated to examining the positive kernel and its Hilbert basis.

To understand the positive kernel it is important to first understand the kernel of the Laplacian. The kernel of the Laplacian is

$$\ker L_i = \ker \partial_{i+1}\partial_{i+1}^t = \ker \partial_{i+1}^t,$$

by a standard result in linear algebra.

This is useful because of the chain property of the boundary maps: im $\partial_i^t \subset \ker \partial_{i+1}^t = \ker L_i$, and the image of $\partial_i^t$ is easier to understand in general. In fact, we can give a combinatorial description of a generating set:

**Definition 2.14.** The **star** $S_f$ at an $i-1$-dimensional face $f = \overline{v_0 v_1 \cdots v_{i-1}}$ of $C$ is defined as $\partial_i^t(f)$. It takes the following form:

$$S_f = \sum_{\overline{v_0 v_1 \cdots v_k v v_{k+1} \cdots v_{i-1}} \in C} (-1)^{k+1} \overline{v_0 v_1 \cdots v_k v v_{k+1} \cdots v_{i-1}}$$

The sum is over all $i$-dimensional faces of $C$ that contain $f$, and the sign depends on where the new vertex lies with regards to the order of vertices of $f$.

The name star comes from the fact that it is the sum of all divisors radiating from some face, with varying signs.

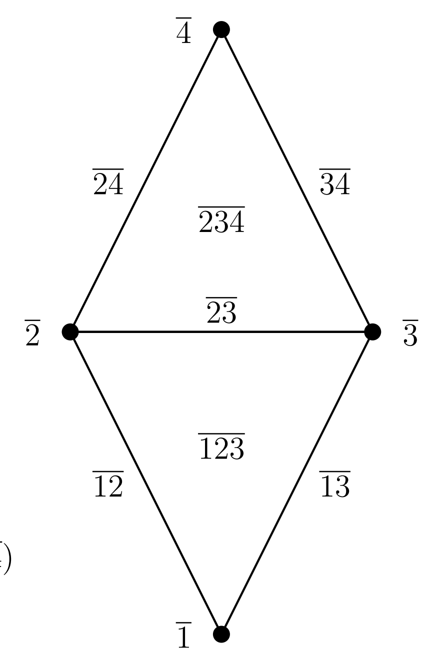
**Example 2.4.** Returning to our example complex, the star $S_{\overline{2}}$ at vertex $\overline{2}$ is

$$S_{\overline{2}} = \overline{12} - \overline{23} - \overline{24}$$

and the star $S_{\overline{23}}$ at edge $\overline{23}$ is

$$S_{\overline{23}} = \overline{234} + \overline{123}.$$

We can also verify that these are both in the kernel of their respective Laplacians, since we know $L_1$, and $L_2 = 0$:




$$\begin{aligned} L_1(S_{\overline{2}}) &= (\overline{12} - \overline{13} + \overline{23}) \\ &\quad - (\overline{12} - \overline{13} + 2 \cdot \overline{23} - \overline{24} + \overline{34}) - (-\overline{23} + \overline{24} - \overline{34}) \\ &= 0. \end{aligned}$$

These stars generate im $\partial_i^t$, and all lie in the kernel of the Laplacian. They also allow us to prove a key fact about the kernel of the Laplacian:

**Proposition 2.2.** *There exists an element* $p = (p_1, \ldots, p_n)$ *in* $\ker L_i$ *such that* $p_j$ *is strictly positive for all* $j$.

*Proof.* Assume for contradiction that no such element exists. Then for every element $D$ in $\ker L_i$, let $m$ denote the least (in lexicographic ordering) $i$-dimensional face such that $D(m) \leq 0$.

Let $D_{max}$ be an element of the kernel that maximizes $m$. Let $\overline{v_0 \cdots v_i}$ be this maximal $m$. Consider the star $S = S_{\overline{v_1 v_2 \cdots v_i}}$. The coefficient of $m$ in $S$ is 1, and if $m'$ is another $i$-dimensional face such that $m' < m$, then $m'$ begins with a vertex smaller than $v_1$ meaning one of the two cases occurs: either $m' = \overline{v' v_1 v_2 \cdots v_n}$, in which case the coefficient of $m'$ in $S$ is 1, or $m'$ does not contain $\overline{v_1 v_2 \cdots v_i}$ at all, and the coefficient of $m'$ in $S$ is 0. Either way, if $m' < m$, the coefficient of $m'$ in $S$ is nonnegative.

Now consider $D' = D_{max} + (1 - D_{max}(m))S$. This is an element of the kernel, and $D'(f) > 0$ for all faces $f \leq m$. But this contradicts the maximality of $m$, so our initial assumption must be false. □

**Corollary 2.1.** The $\mathbb{Z}$-span of the positive kernel is $\ker L_i$. Consequently, the orthogonal complement of the positive kernel is the same as $(\ker L_i)^{\perp}$.

*Proof.* Let $p$ be as in Proposition 2.2. Then we can extend $\{p\}$ to a basis for $\ker L_i$, $\{p, \beta_2, \beta_3, \ldots, \beta_k\}$. For any integer $N$,

$$\{p, \beta_2 + pN, \beta_3 + pN, \ldots, \beta_k + pN\}$$

is also a basis for $\ker L_i$. If $N$ is sufficiently large,

$$\{p, \beta_2 + pN, \beta_3 + pN, \ldots, \beta_k + pN\}$$

will be contained within the positive kernel, implying the result.

□

The stars can also be used to concretely construct elements of the positive kernel, as shown in the following theorem.

**Theorem 2.2.** *Let* $C$ *be a simplicial complex, and let* $f_1, f_2, \ldots, f_n$ *be the lexicographic ordering of its* $(i-1)$*-dimensional faces. Then the following set* $S$ *of elements all lie* *in the $i^{th}$ positive kernel:*

$$(-1)^i S_{f_1}$$
$$(-1)^i (S_{f_1} + S_{f_2})$$
$$(-1)^i (S_{f_1} + S_{f_2} + S_{f_3})$$
$$(-1)^i (S_{f_1} + S_{f_2} + ... + S_{f_4})$$
$$\vdots$$
$$(-1)^i (S_{f_1} + S_{f_2} + ... + S_{f_n})$$

*Furthermore,*

1. *If $H^i = \ker \partial^t_{i+1} / im\ \partial^t_i$ is trivial, $\ker L_i = span\ (S)$.*
2. *If $i = 1$ and $H^1$ is trivial, then the first $n-1$ elements of $S$ form a basis for $\ker L_i$.*
3. *If $i = 1$, $H^1$ is trivial, and for each vertex $\overline{j}$ except $\overline{1}$, $\overline{(j-1)j}$ is an edge of $C$, then this is the unique minimal Hilbert basis of the positive kernel.*

*Proof.* Since each star lies in $\ker L_i$, each sum also lies in $\ker L_i$, so we need only to show each sum has only nonnegative coefficients.

Consider the coefficient of $\overline{v_0 v_1 \cdots v_i}$ in $(-1)^i (S_{f_1} + S_{f_2} + ... + S_{f_t})$. The stars that contribute are those that correspond to subfaces of $\overline{v_0 v_1 \cdots v_i}$.

The first subface to appear in the lexicographic ordering is $\overline{v_0 v_1 \cdots v_{i-1}}$, in which the coefficient of $\overline{v_0 v_1 \cdots v_i}$ is $(-1)^i$. The second subface to appear is $\overline{v_0 v_1 \cdots v_{i-2} v_i}$, in which the coefficient of $\overline{v_0 v_1 \cdots v_i}$ is $(-1)^{i-1}$. This pattern continues, with the sign of the coefficient contributed by the star corresponding to the next subface flipping each time.

Therefore, the coefficient of $\overline{v_0 v_1 \cdots v_i}$ in $(-1)^i (S_{f_1} + S_{f_2} + ... + S_{f_t})$ is simply

$$(1 - 1 + 1 - 1 + 1 - \cdots \pm 1) = 1 \text{ or } 0$$

so each $(-1)^i (S_{f_1} + S_{f_2} + ... + S_{f_t})$ is in the positive kernel.

If $H^i$ is trivial, then im $\partial^t_i = \ker \partial^t_{i+1} = \ker L_i$, and $S$ spans im $\partial^t_i$, since the stars do, so $S$ spans $\ker L_i$.

If $i = 1$, then consider any linear combination $a_1 S_{f_1} + \cdots + a_n S_{f_{n-1}}$ of the first $n-1$ stars. Its boundary is

$$\partial_1 (a_1 S_{f_1} + \cdots + a_n S_{f_{n-1}}) = \partial_1 \partial^t_1 (a_1 f_1 + \cdots a_{n-1} f_{n-1}) = L_0 (a_1 f_1 + \cdots a_{n-1} f_{n-1})$$

By Proposition 1.1, this is nonzero, and therefore $a_1 S_{f_1} + \cdots + a_n S_{f_{n-1}}$ is nonzero. Therefore the first $n-1$ elements are linearly independent. Additionally, considering the coefficient of each face, we can see that $-(S_{f_1} + S_{f_2} + ... + S_{f_n}) = 0$. If $H^1$ is trivial, $S$ also spans $\ker L_i$, and the first $n-1$ elements of $S$ therefore form a basis of $\ker L_i$.

Now assume $H^1$ is trivial and $i = 1$, and that for each vertex $\overline{j}$ except $\overline{1}$, $\overline{(j-1)j}$ is an edge of $C$.

The coefficient of $\overline{(j-1)j}$ in $S_{\overline{j-1}}$ is $-1$, in $S_{\overline{j}}$ is 1, and in all other 1-dimensional stars is 0. Therefore, the only element of $S$ with nonzero coefficient of $\overline{(j-1)j}$ is $-(S_{\overline{1}} + S_{\overline{2}} + ... + S_{\overline{j-1}})$, which has coefficient 1.

Now by previous results, if $x$ is in the positive kernel, $x$ can be written as

$$x = a_1(-S_{\overline{1}}) + a_2(-S_{\overline{1}} - S_{\overline{2}}) + + \cdots + a_{n-1}(-S_{\overline{1}} - S_{\overline{2}} - \cdots - S_{\overline{n-1}}).$$

Then the coefficient of $\overline{(j-1)j}$ in $x$ is $a_{j-1}$, so $a_{j-1} \geq 0$, and therefore we have actually written $x$ as a nonnegative linear combination in the first $n-1$ elements of $S$, so the first $n-1$ elements of $S$ form a Hilbert basis for the positive kernel. This Hilbert basis is also minimal, because the first $n-1$ elements of $S$ are linearly independent. □

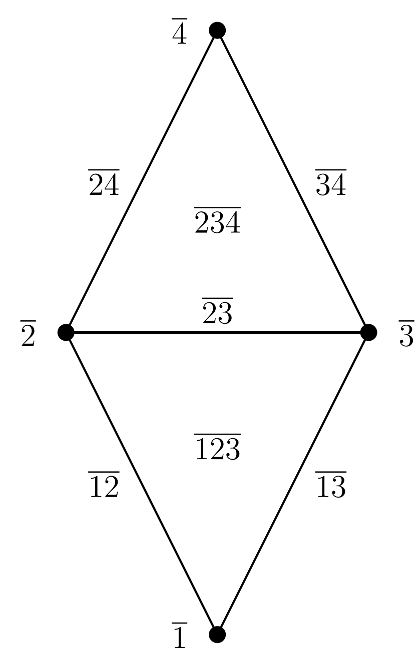
**Example 2.5.** Our example complex happens to satisfy all of the hypotheses needed for this theorem to give a Hilbert basis: $\overline{12}, \overline{23}, \overline{34}$ all are edges of the diamond simplicial complex, and cohomology calculations in topology tell us that $H^1$ is trivial. Computation gives

$$\begin{aligned}
-S_{\overline{1}} &= \overline{12} + \overline{13} \\
-S_{\overline{1}} - S_{\overline{2}} &= \overline{12} + \overline{13} - (\overline{12} - \overline{23} - \overline{24}) \\
&= \overline{13} + \overline{23} + \overline{24} \\
-S_{\overline{1}} - S_{\overline{2}} - S_{\overline{3}} &= \overline{13} + \overline{23} + \overline{24} - (\overline{13} + \overline{23} - \overline{34}) \\
&= \overline{24} + \overline{34}
\end{aligned}$$

Therefore,

$$\begin{gathered}
\overline{12} + \overline{13} \\
\overline{13} + \overline{23} + \overline{24} \\
\overline{24} + \overline{34}
\end{gathered}$$

is the minimal Hilbert basis for the $1^{st}$ positive kernel of the diamond complex.



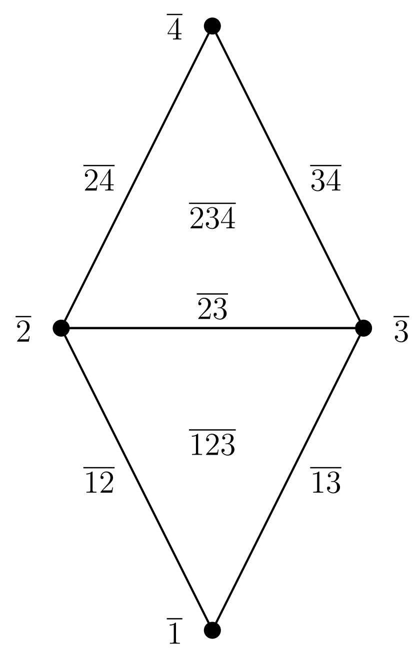
## 2.3 The Greedy Algorithm

One natural question to ask about higher order divisors is whether or not there is a generalization of the greedy algorithm that works for divisors of higher dimension. While we could simply apply the greedy algorithm to a higher-dimensional divisor, it no longer gives accurate results, and may even never terminate.

**Example 2.6.** Examining the diamond simplicial complex again, we can look at the 1-dimensional divisor $(-1, 0, 0, 0, 0)$. If we were to naively apply the 0-dimensional greedy algorithm to this divisor, it might proceed as follows:

1. Start with the divisor $(-1, 0, 0, 0, 0)$.
2. Borrow at edge $\overline{12}$ to obtain the divisor $(0, -1, 1, 0, 0)$.
3. Borrow at edge $\overline{13}$ to obtain the divisor $(-1, 0, 0, 0, 0)$.

But now we are back where we started, and the greedy algorithm will just cycle between these two divisors.




There were three key facts that the proof of the 0-dimensional greedy algorithm relied on that are no longer true for higher dimensions:

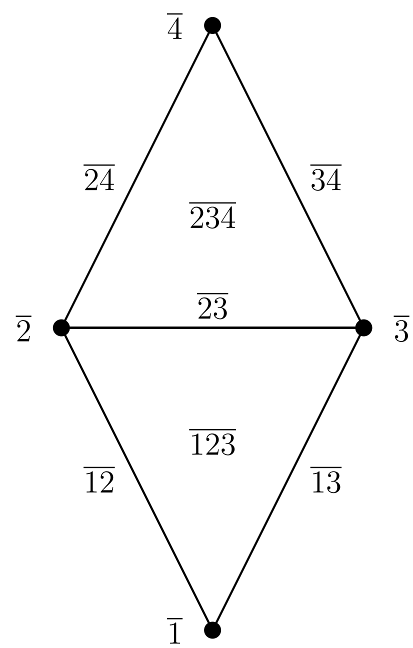
1. The kernel of $L_0$ is span $(1, 1, \ldots, 1)$.
2. The only borrowing move that increases a divisor's value at vertex $v$ is the borrowing move at $v$.
3. The degree of a divisor is invariant under linear equivalence.

The first fact can be resolved through appeal to the Hilbert basis of the positive kernel. Instead of requiring that the greedy algorithm avoids borrowing at all vertices, we can require that the greedy algorithm avoids borrowing at any combination (with multiplicity) of vertices that lies in the Hilbert basis. For example, in the previous example, instead of borrowing at $\overline{13}$, the greedy algorithm would correctly conclude

$(-1,0,0,0,0)$ to be unwinnable, since $\overline{13}+\overline{23}$ lies in the Hilbert basis. This will prevent the greedy algorithm from becoming stuck in a loop as in the previous example, since a sequence of borrowing moves that leaves a divisor unchanged lies in the positive kernel, and some subset of those moves must therefore be in the Hilbert basis. However, even with this modification, it is still possible for this modified greedy algorithm to fail:

**Example 2.7.** Examining the diamond simplicial complex again, this time looking at the divisor $(-2,2,-4,2,-2)$. Our modified greedy algorithm might proceed as follows:

1. Start with the divisor $(-2,2,-4,2,-2)$.
2. Borrow at edge $\overline{12}$ to obtain the divisor $(-1,1,-3,2,-2)$.
3. Borrow at edge $\overline{12}$ to obtain the divisor $(0,0,-2,2,-2)$.
4. Borrow at edge $\overline{23}$ to obtain the divisor $(1,-1,0,1,-1)$.
5. Borrow at edge $\overline{34}$ to obtain the divisor $(1,-1,1,0,0)$.
6. Since $\overline{12}+\overline{13}$ is in the Hilbert basis, we cannot borrow at $\overline{13}$, so the greedy algorithm will determine this divisor to be unwinnable.




But this divisor is winnable, as performing that last borrowing move would show; it is linearly equivalent to 0. Note that the output of the greedy algorithm depends on how we chose at which edge to borrow, when we did have a choice. If we had instead chosen to borrow at $\overline{23}$ twice to begin with, the algorithm would have correctly found the divisor to be winnable.

The issue now is that second fact about 0-dimensional divisors that fails to hold in higher dimensions. It is possible to increase the value of a divisor at a face by borrowing at some other face. By Proposition 2.2, if a divisor is winnable, then, as in the 0-dimensional case, it can be won by only borrowing moves. Additionally, there must be a winning sequence of borrowing moves that avoids anything in the Hilbert basis for the positive kernel. But a borrowing move at any negative vertex is no longer guaranteed to be a part of this winning sequence of borrowing moves, so we can no longer be sure that the greedy algorithm will reach this sequence.

This second fact does not seem to have a simple fix or generalization to divisors of higher dimension, and because of it, neither does the greedy algorithm.

The third fact, that the degree of a divisor was invariant under linear equivalence, and which was used in to demonstrate termination of the 0-dimensional greedy algorithm, does have a nice generalization to higher dimensions. The following section will explore that generalization.

## 2.4 Degrees

The naive generalization for the degree of a $d$-dimensional divisor to be the sum of the coefficients of the $d$-dimensional faces is no longer particularly useful, as it is no longer an invariant of linear equivalence. We instead generalize to a different invariant, which we will call the **degree vector** of a divisor:

**Definition 2.15.** Let $H = \{h_1, ..., h_k\}$ be the Hilbert basis for the positive kernel of $L_d$. Then the **degree vector** of a $d$-dimensional divisor $D$ is

$$\deg(D) = (\langle h_1, D\rangle, \ldots, \langle h_k, D\rangle)$$

where the inner product is the usual inner product with respect to the standard basis.

This definition retains many of the properties that the definition of degree had in the graph case:

**Proposition 2.3.** *The degree vector is invariant under linear equivalence.*

*Proof.* It suffices to show that $\langle L_d x, h_i\rangle = 0$ for any $x \in C_d$ and any $h_i$. But the Laplacian is self-adjoint since $L_d^t = L_d$, so $\langle L_d x, h_i\rangle = \langle x, L_d h_i\rangle = \langle x, 0\rangle = 0$. □

**Corollary 2.2.** A divisor is winnable only if its degree vector lies in the positive orthant.

*Proof.* A winnable divisor $D$ is linearly equivalent to an effective divisor $D'$. Then $\deg(D) = \deg(D')$ and the degree vector of an effective divisor is clearly in the positive orthant. □

Most notably, the fact that any 0-dimensional divisor of sufficiently high degree is winnable also generalizes to higher dimensions. To prove this, we will use the following lemmas:

**Lemma 2.1.** *Let $M \subset \mathbb{R}^n$ be a linear subspace containing an element $y = (y_1, \ldots, y_n)$ such that $y_i > 0$ for each $i = 1, \ldots, n$. Then the following equality of sets holds:*

$$M^\perp + \mathcal{O}^+ = \bigcap_{m \in M \cap \mathcal{O}^+} (m^\perp + \mathcal{O}^+)$$

*where the sum is a Minkowski sum of sets.*

*Proof.* Since $M^\perp$ and $\mathcal{O}^+$ are both convex polyhedra in $\mathbb{R}^n$, so is their sum $M^\perp + \mathcal{O}^+$. Therefore, $M^\perp + \mathcal{O}^+$ can be expressed as the intersection of some number of half-spaces:

$$M^\perp + \mathcal{O}^+ = \bigcap_{i=1}^{l} H_i$$

where each $H_i$ is a half-space with bounding half-plane $P_i$ such that

$$M^\perp + \mathcal{O}^+ \subset H_i \text{ and } (M^\perp + \mathcal{O}^+) \cap P_i \neq \emptyset.$$

Now $(M^\perp + \mathcal{O}^+) \cap P_i$ is a non-empty face of $M^\perp + \mathcal{O}^+$, so it can be expressed as a sum of a non-empty face of $M^\perp$ and a non-empty face of $\mathcal{O}^+$. Every non-empty face of both $M^\perp$ and $\mathcal{O}^+$ contains 0, so $(M^\perp + \mathcal{O}^+) \cap P_i$ also contains 0.

Let $x \in M^\perp$. Since $P_i$ contains 0, either $x \in P_i$ or $x$ and $-x$ lie on opposite sides of $P_i$. But $x$ and $-x$ are both in $M^\perp + \mathcal{O}^+$, so they cannot lie on opposite sides of $P_i$. Therefore $M^\perp \subset P_i$. Taking the orthogonal complement of both sides reverses the containment, and we have $P_i^\perp \subset M$.

Let $p_i = (p_{i,1}, \ldots, p_{i,n}) \in P_i^\perp$. Assume for contradiction that there exists $j, k$ such that $p_{i,j} > 0$ and $p_{i,k} < 0$. Without loss of generality, $j = 1, k = 2$. Then there exists a vector of the form $x = (1 + a, 1 + b, 1, 1, \ldots, 1)$ with $a, b > 0$ such that $x \cdot p_i = 0$ and $x \in P_i$. But for sufficiently small $\epsilon$, Both $x - \epsilon p_i$ and $x + \epsilon p_i$ are in $\mathcal{O}^+$, contradicting the fact that $\mathcal{O}^+$ lies entirely on one side of $P_i$. Therefore $p_i$ or $-p_i$ lies in $\mathcal{O}^+$.

Combining these two facts we can write $P_i = m_i^\perp$ for some $m_i \in M \cap \mathcal{O}^+$. Since $\mathcal{O}^+$ lies entirely on one side of $P_i$, and $\operatorname{span}(\mathcal{O}^+) = \mathbb{R}^n$, we can then write $H_i = m_i^\perp + \mathcal{O}^+$.

We therefore have:

$$M^\perp + \mathcal{O}^+ = \bigcap_{i=1}^{l} (m_i^\perp + \mathcal{O}^+) \supset \bigcap_{m \in M \cap \mathcal{O}^+} (m^\perp + \mathcal{O}^+).$$

where the containment follows because the left hand side is an intersection of a sub-collection of the sets in the right hand side intersection.

The reverse containment is clear, since $M^\perp \subset m^\perp$ for any $m \in M$. $\square$

Taking $M = \ker L_i$ gives the following corollary:

**Corollary 2.3.** There exists a finite set $P$ of $i$-dimensional divisors such that for every $i$-dimensional divisor $D$ with an entirely nonnegative degree vector, there exists some $p \in P$ and some effective divisor $D'$ such that

$$\deg(D) = \deg(D' + p)$$

*Proof.* By Proposition 2.2, $\ker L_i^{\mathbb{R}}$ contains a fully positive element $Y$, so Lemma 2.1 applies to $\ker L_i^{\mathbb{R}}$. We can then intersect with $\mathbb{Z}^n$ to obtain the following,

$$((\ker L_i^{\mathbb{R}})^{\perp} + \mathcal{O}^+) \cap \mathbb{Z}^n = (\bigcap_{m \in \ker L_i^{\mathbb{R}} \cap \mathcal{O}^+} (m^{\perp} + \mathcal{O}^+)) \cap \mathbb{Z}^n.$$

Now both $(\ker L_i^{\mathbb{R}})^{\perp}$ and $\mathcal{O}^+$ are rational polyhedral cones and can thus be written as a Minkowski sum of the integral points they contain and some compact set:

$$(\ker L_i^{\mathbb{R}})^{\perp} = ((\ker L_i^{\mathbb{R}})^{\perp} \cap \mathbb{Z}^n) + P_1$$
$$\mathcal{O}^+ = (\mathcal{O}^+ \cap \mathbb{Z}^n) + P_2$$

for some compact sets $P_1$ and $P_2$.

Putting these together gives

$$\begin{aligned} ((\ker L_i^{\mathbb{R}})^{\perp} + \mathcal{O}^+) \cap \mathbb{Z}^n &= (((\ker L_i^{\mathbb{R}})^{\perp} \cap \mathbb{Z}^n) + (\mathcal{O}^+ \cap \mathbb{Z}^n) + P_1 + P_2) \cap \mathbb{Z}^n \\ &= ((\ker L_i^{\mathbb{R}})^{\perp} \cap \mathbb{Z}^n) + (\mathcal{O}^+ \cap \mathbb{Z}^n) + ((P_1 + P_2) \cap \mathbb{Z}^n) \end{aligned}$$

Taking $P = ((P_1 + P_2) \cap \mathbb{Z}^n)$, which is a finite set since $P_1 + P_2$ is compact,

$$((\ker L_i^{\mathbb{R}})^{\perp} \cap \mathbb{Z}^n) + (\mathcal{O}^+ \cap \mathbb{Z}^n) + ((P_1 + P_2) \cap \mathbb{Z}^n)$$

is the set of all divisors that can be written as $D_0 + D' + p$, where $D_0$ has degree vector 0, $D'$ is effective, and $p \in P$. Alternatively, it is the set of all divisors who have the same degree vector as $D' + p$ for some effective divisor $D'$ and $p \in P$.

Considering the right-hand side of our original equation, for each $m \in \ker L_i^{\mathbb{R}} \cap \mathcal{O}^+$,

$$m^{\perp} + \mathcal{O}^+ = \{D \in \mathbb{R}^n \mid D \cdot m \geq 0\}$$

and therefore

$$(\bigcap_{m \in \ker L_i^{\mathbb{R}} \cap \mathcal{O}^+} (m^{\perp} + \mathcal{O}^+)) \cap \mathbb{Z}^n = \bigcap_{m \in \ker L_i^{\mathbb{R}} \cap \mathcal{O}^+} \{D \in \mathbb{Z}^n \mid D \cdot m \geq 0\}$$

In particular, when $m = h_i$ is an element of the Hilbert basis of the positive kernel, this is the set of divisors whose $i^{th}$ degree is nonnegative. Therefore the right hand side contains only divisors whose degrees are all nonnegative. Additionally, if a divisor $D$ has all nonnegative degrees, then for any $m \in \ker L_i^{\mathbb{R}} \cap \mathcal{O}^+$, we can express $m$ in terms of the Hilbert basis:

$$m = c_1 h_1 + \cdots + c_k h_k$$

for some nonnegative $c_i$. Therefore

$$D \cdot m = c_1(D \cdot h_1) + \cdots + c_k(D \cdot h_k) \geq 0$$

Therefore the right-hand side of our original equality is the set of all divisors with nonnegative degrees, and the conclusion follows. $\square$

We also need another lemma:

**Lemma 2.2.** *Let $C$ be a simplicial complex, and $L_i$ its $i^{th}$ Laplacian. Then im $L_i \subset (\ker L_i)^\perp$ and $(\ker L_i)^\perp/$im $L_i$ is a finite group.*

*Proof.* Consider the linear extension of $L_i$ to $\mathbb{R}^n$, which we will denote by $L_i^{\mathbb{R}} : \mathbb{R}^n \to \mathbb{R}^n$. Then $(\ker L_i)^\perp$ is the set of all integral points in $(\ker L_i^{\mathbb{R}})^\perp$, a lattice in $(\ker L_i^{\mathbb{R}})^\perp$.

Additionally, im $L_i$ is a lattice in im $L_i^{\mathbb{R}}$.

Now consider $L_i^{\mathbb{R}} x$ in the image of $L_i^{\mathbb{R}}$ and $k \in \ker L_i^{\mathbb{R}}$. Then since $L_i^t = (\partial_{i+1}\partial_{i+1}^t)^t = \partial_{i+1}\partial_{i+1}^t = L_i$,

$$\langle L_i^{\mathbb{R}} x, k\rangle = \langle x, (L_i^{\mathbb{R}})^t k\rangle = \langle x, L_i^{\mathbb{R}} k\rangle = \langle x, 0\rangle = 0.$$

So im $L_i^{\mathbb{R}} \subset (\ker L_i^{\mathbb{R}})^\perp$. By rank-nullity, these two spaces have the same dimension, and are thus equal.

Putting everything together, we find that im $L_i$ is a sublattice of $(\ker L_i)^\perp$, and the conclusion follows. $\square$

With this corollary and lemma we can now prove the following theorem:

**Theorem 2.3.** *Let $C$ be a simplicial complex. Then there exists a degree vector $w = (w_1, \ldots, w_k)$ such that every divisor $D$ of $C$ with $\deg(D)_i \geq w_i$ for all $i = 1, \ldots, k$ is winnable.*

*Proof.* Let $P$ be the finite set of divisors as in Lemma 2.1. Since it is finite, $P$ lies within some sphere of radius $r_1 \in \mathbb{Z}$ centered around the origin.

By Lemma 2.2, $(\ker L_d)^\perp/(\text{im } L_d)$ is a finite group. Let $S$ then be a finite set of coset representatives for $(\ker L_d)^\perp/(\text{im } L_d)$ in $(\ker L_d)^\perp$. Since it is finite, $S$ is bounded within some sphere of radius $r_2 \in \mathbb{Z}$ centered around the origin.

Let $D$ be a divisor whose degrees are greater than $\deg(r_1 + r_2, r_1 + r_2, \ldots, r_1 + r_2)$. Then $D - (r_1 + r_2, r_1 + r_2, \ldots, r_1 + r_2)$ is a divisor whose degrees are all nonnegative. By Corollary 2.3, there exists $D'$ effective and $p \in P$ such that $\deg(D' + p) = \deg(D - (r_1 + r_2, r_1 + r_2, \ldots, r_1 + r_2))$. By the definition of $S$, there exists some $s \in S$ such that

$$D - (r_1 + r_2, r_1 + r_2, \ldots, r_1 + r_2) \sim D' + p + s$$

and therefore

$$D \sim D' + p + s + (r_1 + r_2, r_1 + r_2, \ldots, r_1 + r_2).$$

By the definition of $r_1$ and $r_2$, we have that $p + s + (r_1 + r_2, r_1 + r_2, \ldots, r_1 + r_2)$ is an effective divisor, so $D' + p + s + (r_1 + r_2, r_1 + r_2, \ldots, r_1 + r_2)$ is too, and $D$ is winnable.

□

## 2.5 Future Directions

The results discussed in this chapter naturally lead to further questions about simplicial divisors. For example, while we have shown the greedy algorithm and simple modifications to it to fail for simplicial complexes, is there some other efficient algorithm for determining if a divisor is winnable? Currently, algorithms for determining if a simplicial divisor is winnable involve determining if integer points exists inside some polytope which is known to be an NP-hard problem. Is determining if a divisor of an arbitrary graph is winnable also NP-hard, or is there a more efficient algorithm?

Secondly, while we know that there exists some degree vector $w$, such that any divisor with degree vector at least $w$ is winnable, we do not know what $w$ is for an arbitrary simplicial complex as we do for the analogous statement of a graph. Determining what the minimal $w$ is and if there even is a unique minimal $w$ (since degree vectors only have a partial ordering instead of a total ordering), would be an interesting problem to study in the future. Additionally, we do not currently know when $w = 0$ suffices for a simplicial divisor to be winnable. Is it the case that $w = 0$ suffices if and only if our complex is a simplicial tree, as was the case for graphs? Answering this question would be another interesting problem for the future.

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