The Structure of the Jacobian Group of a Graph

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Table of Contents

List of Figures

Abstract

This thesis explores the structure of the Jacobian group of a graph. We begin by defining the structure of finite abelian groups in terms of invariant factors and follow with simple examples of Jacobian groups. We then focus on *how* the number of invariant factors can change when removing edges. In Lorenzini [2008], it is shown that the number of invariant factors changes by at most 1 when an edge is removed from a graph and that if a path graph of $n-3$ edges is removed from the complete graph on n vertices, the resulting graph has cyclic Jacobian group. We determine the size of that cyclic group in terms of Chebyshev polynomials of the second kind. Removing edges tends to decrease the number of invariant factors of the corresponding Jacobian groups. We present a family of graphs such that removal of an edge results in a graph for which the number of invariant factors increases.

Dedication

To Dr. Joe Busch, whos words of mathematical wisom are ingrained in my brain:

"Use what you have to get what you want."

Introduction

To understand the question we wish to answer in this thesis, we first need some intuition on the subject. Let us introduce the dollar game.

Let $G = (V, E)$ be a graph with vertex set V and a multiset of edges, E. Consider the graph G, where there are four vertices, labeled v_1 to v_4:

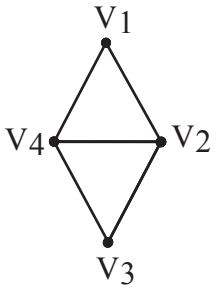

Figure 1: Example of a Labeled Graph

Let each vertex on G represent a person playing the dollar game. Some people have stronger relationships than others. Some people will be wealthier than others, and some will be in debt. We can represent the wealth of each person by assigning numbers to each vertex.

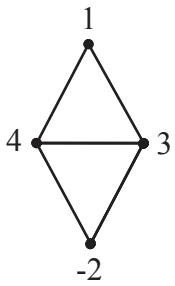

Figure 2: Distribution of Wealth on a Graph

Each person can lend or borrow a dollar along each edge. If someone lends a dollar, they must lend a dollar on each edge to an adjacent person.

Let us play a game as an example:

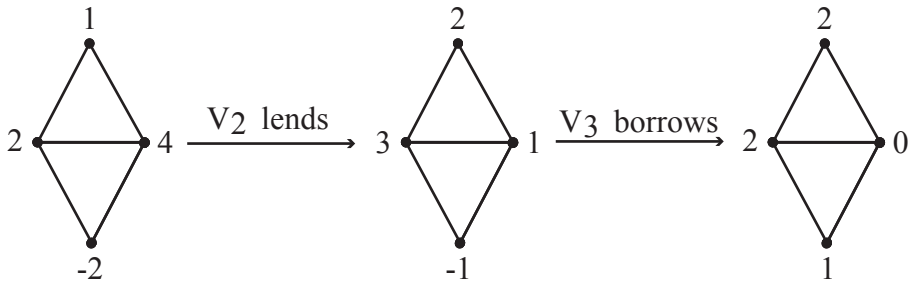

Figure 3: Example Game

Now, everyone is out of debt. If a game can be played such that a sequence of lending and borrowing moves takes everyone out of debt, then the game is said to be *winnable*. We can also play a game where there are multiple edges between players. We show the strength of the relationship by the number of edges between vertices. The rules regarding lending along edges remain the same.

0.1 Divisors on graphs.

We will now discuss the theory behind our game more formally. The distribution of wealth among vertices is called the *divisor* on G. We will use Baker and Norine [2007] as a guide for this section.

Definition 0.1.1. A *divisor*, D, is an element of the free abelian group on the vertices:

$$\mathrm{Div}(G) = \left\{ \sum_{v \in V} a_v v : v \in V \right\}.$$

The *degree* of $D \in \mathrm{Div}(G)$ is $\deg(D) = \sum_{v \in V} D(v)$. We say two divisos $D, D' \in \mathrm{Div}(G)$ are *linearly equivalent*, denoted $D \sim D'$, if D' may be obtained from D by a sequence of lending and borrowing moves.

Example 0.1.1. For instance, in Figure 3, the starting distribution of wealth is $D = v_1 + 4\,v_2 - 2\,v_3 + 2\,v_4$. The ending distribution of wealth is $D' = 2\,v_1 + v_3 + 2\,v_4$ Note that $D \sim D'$, and that $\deg(D) = \deg(D') = 5$.

In general, all divisors in an equivalence class will have the same degree.

Definition 0.1.2. The divisor class for $D \in \mathrm{Div}(G)$ is

$$[D] := \{D' \in \mathrm{Div}(G) : D' \sim D\}$$

and the collection of divisors is called the *Picard group*:

$$\mathrm{Pic}(G) := \mathrm{Div}(G)/\!\sim = \{[D] : D \in \mathrm{Div}(G)\}.$$

We have associated an abelian group, $\mathrm{Pic}(G)$, to every graph:

$$\begin{array}{ccc} \text{Graphs} & \to & \text{Abelian groups} \\ G & \mapsto & \mathrm{Pic}(G). \end{array}$$

The goal of this thesis is to explore the relationship between the structure of G and its group $\mathrm{Pic}(G)$.

Define $\mathrm{Div}^\circ(G)$ to be the set of divisors on G of degree 0. Since the sum of divisors of degree 0 has degree 0, it follows that $\mathrm{Div}^0(G)$ is a subgroup of $\mathrm{Div}(G)$.

Definition 0.1.3. The *Jacobian* of G is

$$\mathrm{Jac}(G) := \mathrm{Div}^\circ(G)/\sim .$$

The $\mathrm{Jac}(G)$ is well-defined because linearly equivalent divisors have the same degree.

Fix $q \in V$. Then there exists an isomorphism

$$\begin{aligned} \mathrm{Pic}(G) &\to \mathbb{Z} \times \mathrm{Jac}(G) \\ D &\mapsto (\deg(D), D - \deg(D)). \end{aligned}$$

Then, to study Pic(G), it suffices to study Jac(G).

From now on, assume the vertices of G are ordered $v_1, \ldots, v_n$. In that case, we make the identification

$$\mathrm{Div}(G) \approx \mathbb{Z}^n$$

$$\sum_{i=1}^{n} a_i v_i \mapsto (a_1, \ldots, a_n).$$

Let the vertices of G be $v_1, \ldots, v_n$. The *adjacency matrix* of G is the $n \times n$ matrix A where

$$A_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

The *degree* of a vertex, denoted $\deg(N)$, is the number of edges that are connected to that vertex.

Definition 0.1.4. Let G be a graph with vertices $v_1, \ldots, v_n$. The *Laplacian* matrix $\Delta = \Delta(G)$ of G, is the $n \times n$ matrix defined as

$$\Delta = D - A,$$

where $D := \mathrm{diag}(\deg(v_i) : i = 1, \ldots, n)$ is the diagonal matrix of vertex degrees, and A is the adjacency matrix of G.

The *reduced Laplacian* matrix with respect to a vertex q is the $(n-1) \times (n-1)$ matrix formed from the Laplacian matrix by removing the row and column corresponding to q. It is denoted $\widetilde{\Delta} = \widetilde{\Delta}(G)$.

We have the following well-known theorem:

Theorem 0.1.5.

$$\begin{aligned} \mathrm{Pic}(G) &\approx \mathbb{Z}^n/\mathrm{im}(\Delta) \\ \sum_{i=1}^{n} a_i v_i &\mapsto (a_1, \ldots, a_n) \end{aligned}$$

and

$$\begin{aligned} \mathrm{Jac}(G) &\approx \mathbb{Z}^{n-1}/\mathrm{im}(\tilde{\Delta}) \\ \sum_{i=1}^{n-1} a_i v_i &\mapsto (a_1, \ldots, a_{n-1}). \end{aligned}$$

By the matrix-tree theorem, we have the following corollary:

Corollary 0.1.6. $|\mathrm{Jac}(G)| =$ *the number of spanning trees of* G.

A consequence of this corollary is that the Jacobian group is finite.

0.2 The structure of a finite abelian group.

This section is based on notes by Perkinson [2014]. What is meant by the "structure" of an abelian group and how is it calculated?

Definition 0.2.1 (Finitely Generated Abelian Groups)**.** An abelian group A is *finitely generated* if there exists $a_1, \ldots, a_n \in A$ such that for all $a \in A$, there exists $k_1, \ldots, k_n \in \mathbb{Z}$ such that $a = \sum_{i=1}^{n} k_i a_i$.

Theorem 0.2.2 (Structure Theorem for Finitely Generated Abelian Groups)**.** *Let A be a finitely generated abelian group. Then there exists a nonnegative integer r and positive integers $d_1, d_2, \ldots, d_n$ such that*

$$A \approx \mathbb{Z}^r \times \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_n}.$$

The number r is called the *rank* of A. The d_i are unique if we require d_i to divide d_{i+1} for all i. In that case, the d_i are called the *invariant factors* of A.

The "structure" of a finitely generated group is given by its rank and its invariant factors. We will now sketch a proof of the structure theorem:

Sketch of proof. Say A is an abelian group generated by $a_1, a_2, \ldots, a_n \in A$. Then there is a surjective homomorphism

$$\begin{array}{rcl} \Phi : \mathbb{Z}^n & \twoheadrightarrow & A \\ l_i & \to & a_i. \end{array}$$

Therefore, there is an isomorphism

$$\mathbb{Z}^n / \ker \Phi \xrightarrow{\approx} A.$$

It turns out that the kernel of Φ must be finitely generated.

Say that $g_1, g_2, \ldots, g_m \in \mathbb{Z}^n$ are generators. Let M be the $m \times n$ matrix whose columns are $g_1, g_2, \ldots, g_m$. Then $\mathbb{Z}^n/\text{image}(M) \xrightarrow{\approx} A$. We determine the invariant factors of A by computing the *Smith Normal Form* of M. □

Definition 0.2.3. An $m \times n$ integer matrix is in *Smith Normal Form* if it has the form:

$$\begin{pmatrix} d_1 & & & \\ & d_2 & & 0 \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix}$$

with $d_1 \mid d_2 \mid \ldots$, i.e., d_i evenly divides d_{i+1} for all i.

Definition 0.2.4. An integer row (resp. column) operation or an integer matrix is one of the following:

(1) Swapping two rows (resp. columns)

(2) negating a row (resp. column)

(3) adding one row (resp. column) to another.

Write $M \sim N$ for integer matrices M and N if one can be obtained from the other via a sequence of integer row and column operations. Then $\sim$ is an equivalence operation.

Theorem 0.2.5. *Each equivalence class of $m \times n$ matrices under $\sim$ has a unique matrix in Smith Normal Form.*

Proof. We first show existence by describing an algorithm. Let $M = (m_{ij})$, an $m \times n$ matrix. If $M = 0$, we are done. Otherwise:

(1) By permuting rows and columns, we may assume that m_{11} is the smallest nonzero entry in absolute value. By adding multiples of the first row to other rows or the first column to other columns, attempt to make all entries in the first column and first row except the $(1,1)$-entry equal to zero.

If during the process, a nonzero matrix entry appears with smaller absolute value than m_{11}, you may permute the rows and columns in order to bring that entry to the

$(1,1)$-position. Since the $(1,1)$-entry is nonzero and descending in magnitude, the process eventually terminates with a matrix of the form:

$$\begin{bmatrix} \delta_{11} & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & M' & \\ 0 & & & \end{bmatrix},$$

where M' is a $(m-1) \times (n-1)$ matrix.

(2) If there is any entry of M' that is not divisible by (1,1)-entry, then add the column of that entry to column 1 and go back to step (1). Again, since the $(1,1)$-entry decreases in magnitude, this new process stops, delivering a matrix of form M' such that all the entries of M' are divisible by (1,1)-entry.

(3) Apply steps (1) and (2) now to M' and thus, by recursion, we get a matrix equivalent to M but in Smith Normal Form. If necessary, multiply by -1 to make the diagonal entries nonnegative. □

Note: Let $S = I_m$ and $T = I_n$ be identity matrices, consider the sequence of elementary row operations leading from M to its Smith Normal Form. Whenever a row operation is made, perform the same operation on S and whenever a column operation is performed, perform the same operation on T. In this way, S and T are transformed into matrices U and V, respectively, such that UMV is the Smith Normal Form for M.

Now that we have discussed the structure of finite abelian groups, we can restate the goal of this thesis more carefully. We begin by presenting some simple examples of the Jacobian group for a graph in Chapter 1. In the next chapter, we focus on the invariant factors of the Jacobian and how the number of nontrivial invariant factors can change. We use Lorenzini [2008] in Theorem 2.0.9 of Chapter 2 to show that the number of invariant factors changes by at most 1 when an edge is removed. We then find in Theorem 2.0.10 that the size of the cyclic Jacobian group is $n\,U_{n-3}(\frac{n-2}{2})$ when a path graph of $n-3$ edges are removed from K_n, where U_n is the Chebyshev polynomial of the second kind. In the second section of Chapter 2, we present new properties of the graphs of interest. We have mapped out all subgraphs of K_5 to show instances where an edge is removed and the number of nontrivial invariant factors increases. Lastly, in Theorem 2.1.1 we present a family of graphs such that removal of an edge increases the number of invariant factors.

Chapter 1

First Examples

This chapter will present two examples as a warm-up.

The *complete graph* K_n is the graph with n vertices and such that every pair of vertices forms an edge.

Theorem 1.0.6. $\mathrm{Jac}(K_n) \approx (\mathbb{Z}_n)^{n-2}$.

Proof. To show $\mathrm{Jac}(K_n) \approx (\mathbb{Z}_n)^{n-2}$, we must compute the Smith Normal Form of the reduced Laplacian for K_n, thus obtaining the invariant factors for $\mathrm{Jac}(G)$. The reducedLaplacian matrix is for K is the $(n-1) \times (n-1)$ matrix:

$$\tilde{\Delta} = \begin{bmatrix} n-1 & -1 & \dots & -1 \\ -1 & n-1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & n-1 \end{bmatrix}.$$

The Smith Normal form can be obtained by the following process:

Add columns 2 through $n-1$ to column 1:

$$\begin{bmatrix} 1 & -1 & \dots & -1 \\ 1 & n-1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & -1 & \dots & n-1 \end{bmatrix}.$$

Next, add column 1 to columns 2 through $n-1$:

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & n \end{bmatrix}.$$

Lastly, subtract row 1 from all other rows.

$$\tilde{\Delta} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & n \end{bmatrix}.$$

We have now obtained the invariant factors, and the result follows. □

Let K_n^* be the complete graph K_n with one edge removed; by symmetry it does not matter which edge is removed.

Theorem 1.0.7. $\mathrm{Jac}(K_n^*) \approx \mathbb{Z}_n^{(n-4)} \times \mathbb{Z}_{n(n-2)}$.

Proof. The $n \times n$ Laplacian matrix for K_n^* is as follows:

$$\Delta = \begin{bmatrix} n-1 & -1 & -1 & \dots & -1 \\ -1 & n-1 & -1 & \dots & -1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -1 & -1 & \dots & n-2 & 0 \\ -1 & -1 & \dots & 0 & n-2 \end{bmatrix}.$$

So,

$$\tilde{\Delta} = \begin{bmatrix} n-1 & -1 & -1 & \dots & -1 \\ -1 & n-1 & -1 & \dots & -1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -1 & -1 & \dots & n-2 & 0 \\ -1 & -1 & \dots & 0 & n-2 \end{bmatrix},$$

where $\tilde{\Delta}$ is of size $(n-1) \times (n-1)$. The Smith Normal form can be obtained by the same process as above. After some reduction, we arrive at the matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & n & 0 & 0 & \dots & 0 \\ 0 & 0 & n & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & n-1 & 1 \\ 0 & 0 & 0 & \dots & 1 & n-1 \end{bmatrix}.$$

Let us examine the bottom right 2×2 corner of this matrix:

$$\begin{bmatrix} n-1 & 1 \\ 1 & n-1 \end{bmatrix}.$$

By integer row and column operations, this reduces to:

$$\begin{bmatrix} 0 & 1-(n-1)^2 \\ 1 & 0 \end{bmatrix}.$$

Which further reduces to:

$$\begin{bmatrix} 1 & 0 \\ 0 & n(n-2) \end{bmatrix}.$$

Hence, $\mathrm{Jac}(K_n^*) \approx \mathbb{Z}_n^{(n-4)} \times \mathbb{Z}_{n(n-2)}$. □

Chapter 2

Lorenzini's Theorem and Applications

In this chapter we will discuss the minimal number of edges we must remove from K_n so that the resulting graph has cyclic Jacobian group. These concepts are discussed in Lorenzini's paper, *Smith normal form and Laplacians*, Lorenzini [2008].

To use Lorenzini's method, we need the following theorem (cf. Jacobson [1985]):

Theorem 2.0.8. *Let A be an $m \times n$ integer matrix of rank r. For each $i \leq r$, let A_i be the GCD of the i-minors of A. Then the invariant factors for A are*

$$d_1 = A_1,\ d_2 = A_2/A_1, \ldots,\ d_r = A_r/A_{r-1}.$$

Example 2.0.1. We will show that the Jacobian group of a cycle graph is cyclic.

Proof. Let C_n be the cycle graph with n vertices. Then the reduced laplacian is

$$\tilde{\Delta} = \begin{bmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ 0 & & & -1 & 2 \end{bmatrix}$$

.

We remove the first row and the last column, which gives us

$$\begin{bmatrix} -1 & 2 & -1 & & 0 \\ & -1 & 2 & \ddots & \\ & & -1 & \ddots & -1 \\ & & & \ddots & 2 \\ 0 & & & & -1 \end{bmatrix}$$

.

Clearly, the determinant of this matrix is ± 1 because the matrix is upper triangular and has -1s on the diagonal. Hence, the GCD of the $(n-2)$-minors is 1. Therefore, the group is cyclic. By the Matrix-Tree theorem, the order of the group is the number of spanning trees, which is clearly n. Hence, the group is $\mathbb{Z}_n$. □

Theorem 2.0.9 (Lorenzini [2008])**.** *Let G be a graph, and let G' be the graph obtained by removing one edge from G. Suppose that G and G' are connected. Then the number of invariant factors $\neq 1$ of the reduced Laplacian for G and that for G' differ by ± 1, if at all.*

Proof. After removing a sink vertex from G, order the vertices $v_1, \cdots, v_n$. Assume the removed edge is between v_1 and v_2. Clearly, the reduced Laplacians $\tilde{\Delta} := \tilde{\Delta}(G)$ and $\tilde{\Delta}' := \tilde{\Delta}(G')$ are related because they have many minors in common:

$$\tilde{\Delta} = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} & \dots \\ \ell_{21} & \ell_{22} & \ell_{23} & \dots \\ \ell_{31} & \ell_{32} & \ell_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad \tilde{\Delta}' = \begin{bmatrix} \ell_{11}-1 & \ell_{12}+1 & \ell_{13} & \dots \\ \ell_{21}+1 & \ell_{22}-1 & \ell_{23} & \dots \\ \ell_{31} & \ell_{32} & \ell_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Let $r_1, \cdots, r_n$ be the rows of $\tilde{\Delta}$. Thus,

$$\begin{aligned} r_1' &= r_1 - e_1 + e_2, \\ r_2' &= r_2 + e_1 - e_2, \\ r_i' &= r_i, \quad \text{for } i > 2, \end{aligned}$$

are the rows of $\tilde{\Delta}'$.

To find the k-minors of $\tilde{\Delta}'$, for each choice of k rows, $r_{i_1}, \cdots, r_{i_k}$, we compute $r_{i_1} \wedge \cdots \wedge r_{i_k} \in \bigwedge \mathbb{R}^n$ in terms of the standard basis $\{e_{j_1} \wedge \cdots \wedge e_{j_k}\}_{1 \leq j_1 < \cdots < j_k \leq n}$ for $\bigwedge^k \mathbb{R}^n$. The coefficient in front of $e_{j_1} \wedge \cdots \wedge e_{j_k}$ is the k-minor formed from the rows $i_1, \dots, i_k$ and columns $j_1, \dots, j_k$. We claim that

$$\mathrm{Span}_{\mathbb{Z}}\{k\text{-minors of } \tilde{\Delta}'\} \subseteq \mathrm{Span}_{\mathbb{Z}}\{(k-1)\text{-minors of } \tilde{\Delta}\},$$

hence $\tilde{\Delta}_{k-1} | \tilde{\Delta}_k'$.

For instance, consider the case $i_1 = 1, i_2 = 2$, and $3 \leq i_3 < \cdots < i_k \leq n$. Let $\omega = r'_3 \wedge \cdots \wedge r'_k = r_3 \wedge \cdots \wedge r_k$. Then

$$\begin{aligned} r'_1 \wedge r'_2 \wedge \omega &= (r_1 - e_1 + e_2) \wedge (r_2 + e_1 - e_2) \wedge \omega \\ &= r_1 \wedge r_2 \wedge \omega - e_1 \wedge r_1 \wedge \omega + e_2 \wedge r_1 \wedge \omega \\ &\quad - e_1 \wedge r_2 \wedge \omega + e_1 \wedge e_2 \wedge \omega + e_2 \wedge r_2 \wedge \omega + e_2 \wedge e_1 \wedge \omega. \end{aligned}$$

The coefficients of $r_1 \wedge r_2 \wedge \omega$ are k-minors of $\tilde{\Delta}$, and hence are contained in the span of the $(k-1)$-minors of $\tilde{\Delta}$. The nonzero coefficients of

$$e_1 \wedge r_1 \wedge \omega, \quad e_2 \wedge r_1 \wedge \omega, \quad e_1 \wedge r_2 \wedge \omega, \quad e_2 \wedge r_2 \wedge \omega$$

are $(k-1)$-minors of $\tilde{\Delta}$. Finally, since $e_1 \wedge e_2 \wedge \omega + e_2 \wedge e_1 \wedge \omega = 0$, we do not need to consider $(k-2)$-minors. This handles the case of the k-minors that involve r'_1 and r'_2. The cases that do not involve both r'_1 and r'_2 have a similar arguments.

Writing the r_is in terms of r'_is, and repeating the argument shows that

$$\mathrm{Span}_{\mathbb{Z}}\{k\text{-minors of } \tilde{\Delta}\} \subseteq \mathrm{Span}_{\mathbb{Z}}\{(k-1)\text{-minors of } \tilde{\Delta}'\},$$

hence $\tilde{\Delta}'_{k-1} | \tilde{\Delta}_k$. Suppose $\tilde{\Delta}_k = 1$. Then since $\tilde{\Delta}'_{k-1} | \tilde{\Delta}_k$, we have $\tilde{\Delta}'_{k-1} = 1$, which implies $\tilde{\Delta}'_i = 1$ for $i \leq k-1$. Similarly, $\tilde{\Delta}'_k = 1$ implies $\tilde{\Delta}_i = 1$ for $i \leq k-1$. The result follows. □

Recall the definition of the Chebyshev polynomial of the second kind:

$$U_n(x) = \det \begin{bmatrix} 2x & 1 & & & \\ 1 & 2x & 1 & & 0 \\ & 1 & 2x & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & 1 & 2x \end{bmatrix},$$

where n is the size of the matrix.

Theorem 2.0.10. *Let G be a graph formed by removing a path graph with $n-3$ edges from K_n. Then $\mathrm{Jac}(G)$ is cyclic of size $n\, U_{n-3}(\frac{n-2}{2})$, where U is the Chebyshev polynomial of the second kind.*

Proof. First, we will prove that this group is cyclic. Consider the Laplacian of G:

$$\Delta = \begin{bmatrix} n-2 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & n-3 & 0 & -1 & \vdots & \vdots & \vdots \\ -1 & \ddots & \ddots & \ddots & -1 & -1 & -1 \\ \vdots & \ddots & 0 & n-3 & 0 & -1 & -1 \\ -1 & \dots & -1 & 0 & n-2 & -1 & -1 \\ -1 & \dots & -1 & -1 & -1 & n-1 & -1 \\ -1 & \dots & -1 & -1 & -1 & -1 & n-1 \end{bmatrix}.$$

Subtract the last row from all other rows to get:

$$\begin{bmatrix} n-1 & 1 & 0 & 0 & 0 & 0 & -n \\ 1 & n-2 & 1 & 0 & 0 & 0 & -n \\ 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & 1 & n-2 & 1 & 0 & -n \\ 0 & \dots & 0 & 1 & n-1 & 0 & -n \\ 0 & \dots & 0 & 0 & 0 & n & -n \\ -1 & \dots & -1 & -1 & -1 & -1 & n-1 \end{bmatrix}.$$

Next, subtract the last row from row $n-2$:

$$\begin{bmatrix} n-1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & n-2 & 1 & 0 & 0 & 0 & -n \\ 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & 1 & n-2 & 1 & 0 & -n \\ 1 & \dots & 1 & 2 & n & 1 & -2n+1 \\ 0 & \dots & 0 & 0 & 0 & n & -n \\ -1 & \dots & -1 & -1 & -1 & -1 & n-1 \end{bmatrix}.$$

Choosing the last vertex as the sink vertex, we can now look at the $(n-2)$-minor, where the last two rows and the first and last columns of Δ are removed:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ n-2 & 1 & 0 & 0 & 0 & 0 \\ 1 & n-2 & 1 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & n-2 & 1 & 0 \\ 1 & \cdots & 1 & 2 & n & 1 \end{bmatrix}.$$

It is clear that the determinant of this matrix is 1. Hence, the GCD of the determinants of the $(n-2)$-minors is 1, and thus the group is cyclic.

We compute the size of the cyclic group, following a hint from Lorenzini's paper. Let J be the $n \times n$ matrix such that all entries are 1. Then the number of spanning trees of G is $\det(\Delta + J)/n^2$. (For a proof, see Biggs [1993].) By Corollary 0.1.6, the number of spanning trees is $|\operatorname{Jac}(G)|$, the order of our cyclic group. Now,

$$\Delta + \mathrm{J} = \begin{bmatrix} n-1 & 1 & & & & & & & \\ 1 & n-2 & 1 & & & & & & 0 \\ & 1 & n-2 & 1 & & & & & \\ & & & \ddots & & & & & \\ & & & & 1 & n-2 & 1 & & \\ 0 & & & & & 1 & n-1 & 0 & 0 \\ & & & & & & 0 & n & 0 \\ & & & & & & 0 & 0 & n \end{bmatrix}.$$

Removing the last two rows and columns, define A as follows:

$$A = \begin{bmatrix} n-1 & 1 & & & & & \\ 1 & n-2 & 1 & & & & 0 \\ & 1 & n-2 & \ddots & & & \\ & & \ddots & \ddots & 1 & & \\ & & & 1 & n-2 & 1 & \\ 0 & & & & 1 & n-1 & \end{bmatrix}.$$

It follows that $\det(A) = \det(\Delta + J)/n^2 = |\operatorname{Jac}(G)|$.

Note that

$$U_{n-2}\left(\frac{n-2}{2}\right) = \det \begin{bmatrix} n-2 & 1 & & & & & \\ 1 & n-2 & 1 & & & & 0 \\ & 1 & n-2 & \ddots & & & \\ & & \ddots & \ddots & 1 & & \\ & & & 1 & n-2 & 1 & \\ 0 & & & & 1 & n-2 & \end{bmatrix}.$$

So, letting r_i denote the i-th row of the above matrix,

$$\begin{aligned} \det(A) &= \det(e_1 + r_1, r_2, \ldots, r_{n-3}, e_{n-2} + r_{n-2}) \\ &= \det(e_1, r_2, \ldots, r_{n-3}, e_{n-2}) + \det(e_1, r_2, \ldots, r_{n-3}, r_{n-2}) \\ &\qquad + \det(r_1, r_2, \ldots, r_{n-3}, r_{n-2}) + \det(r_1, r_2, \ldots, r_{n-3}, e_{n-2}) \\ &= U_{n-2}\left(\frac{n-2}{2}\right) + 2\,U_{n-3}\left(\frac{n-2}{2}\right) + U_{n-4}\left(\frac{n-2}{2}\right). \end{aligned}$$

It is an easy exercise to show $U_n(x) = 2\,x\,U_{n-1}(x) - U_{n-2}(x)$ by expanding the determinant along rows and columns. Using this identity,

$$\begin{aligned}\det(A) &= U_{n-2}\left(\frac{n-2}{2}\right) + 2\,U_{n-3}\left(\frac{n-2}{2}\right) + (n-2)\,U_{n-3}\left(\frac{n-2}{2}\right) - U_{n-2}\left(\frac{n-2}{2}\right) \\ &= n\,U_{n-3}\left(\frac{n-2}{2}\right).\end{aligned}$$

□

Corollary 2.0.11. *Let G be a connected graph on n vertices such that* $\mathrm{Jac}(G)$ *is cyclic. Then the number of edges, $|E| \leq \binom{n}{2} - (n-3)$ and this inequality is* sharp*, meaning there will be cases such that $|E| = \binom{n}{2} - (n-3)$.*

2.1 New Examples

We expect to see the number of invariant factors go down by 1, if at all, when removing an edge because as more edges are removed, the graph becomes more trivial. We are interested in the cases where an edge is removed, but the number of invariant factors increases.

Figure 2.1 shows every connected subgraph of the complete graph, K_5, up to isomorphism, with the invariant factors for their Jacobian groups.

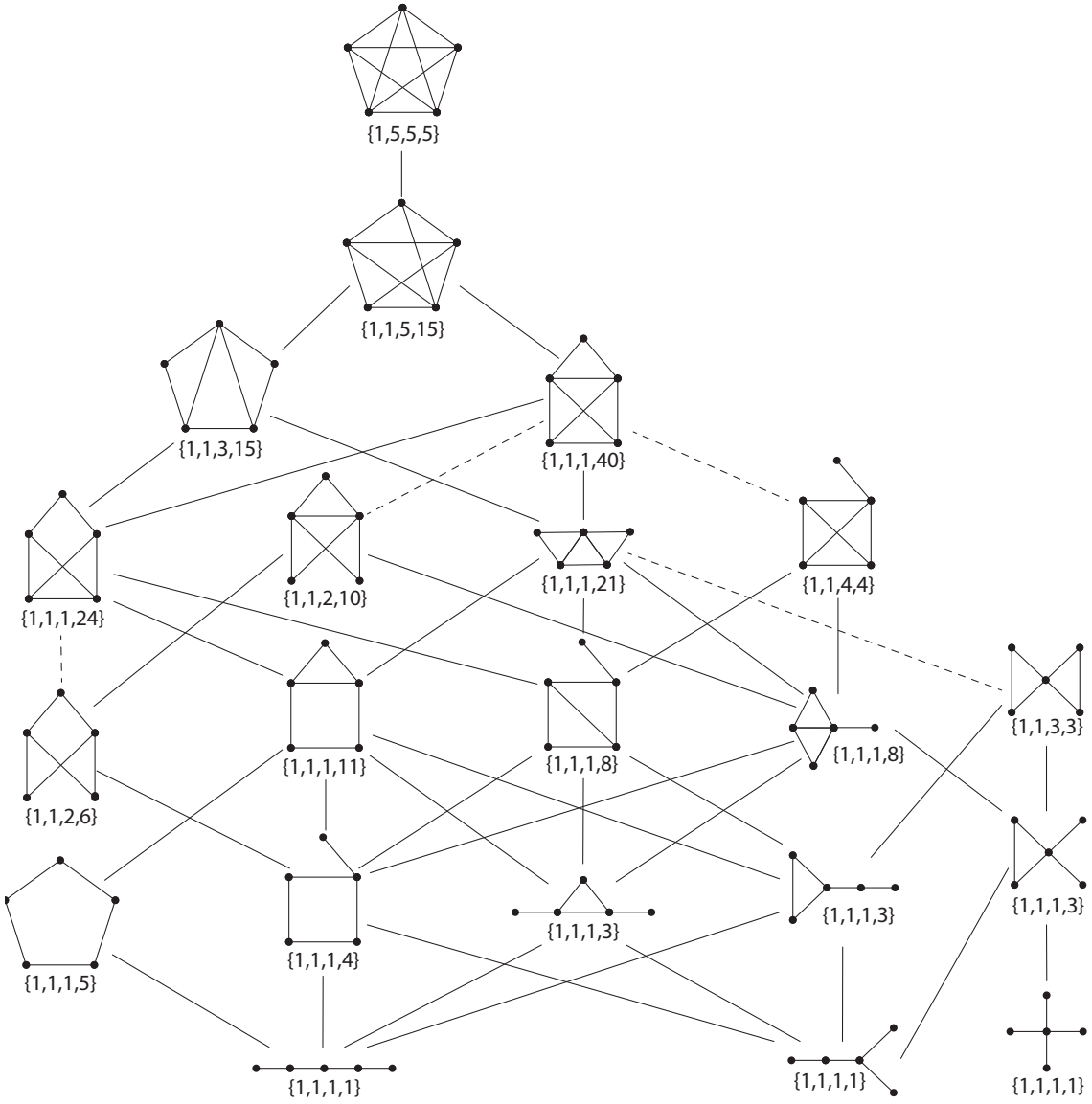

Figure 2.1: Invariant Factors for All Connected Subgraphs of K_5.

The lines between graphs represent graphs that differ by only one edge. Note that dashed lines between graphs indicate that the number of invariant factors increased by one, when removing an edge from one of the graphs.

We now present a family of graphs for which the removal of an edge may increase the number of nontrivial factors.

Theorem 2.1.1. *Let C_m be the cycle graph with m vertices, labeled $u_0, \cdots, u_{m-1}$, and let C_n be the cycle graph with n vertices, labeled $v_0, \cdots, v_{n-1}$. Let H be the graph formed from C_m and C_n by identifying the vertices u_0 and v_0 then adding an edge between u_{m-1} and v_1. Then* $\mathrm{Jac}(H)$ *is cyclic of order $3mn - m - n$. (See Figure 2.2 for the case $m = 6$, $n = 5$.)*

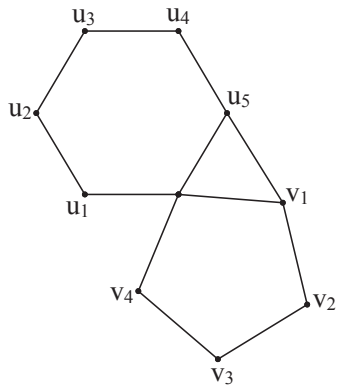

Figure 2.2: C_6 and C_5 Connected By a Sink Vertex and an Edge

Proof. If we order the vertices in the same way as in Figure 2.2, then the reduced laplacian of H will be

$$\tilde{\Delta} = \begin{bmatrix} 2 & -1 & & & & & & 0 \\ -1 & 2 & -1 & & & & & \\ & & \ddots & & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 3 & -1 & & \\ & & & & -1 & 3 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & & \ddots \\ 0 & & & & & & & -1 \quad 2 \end{bmatrix}$$

.

We remove the first row and the last column, which gives us

$$\begin{bmatrix} -1 & 2 & -1 & & & & & 0 \\ & \ddots & & & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 3 & -1 & & \\ & & & & -1 & 3 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & \ddots & \\ 0 & & & & & & & -1 \end{bmatrix}$$

.

By the same argument in Example 2.0.1, the group is cyclic. The order of this group will be the number of spanning trees. To find the number of spanning trees for H, we break the problem down into cases. If the added edge $\{u_{m-1}, v_1\}$ is not included, then there will be a tree for every edge removed for each cycle. Hence, there are mn spanning trees. If the added edge is included, one of $\{u_0, u_{m-1}\}$ or $\{u_0, v_1\}$ are removed, or both are removed. Removing only one of these edges gives $2(m-1)(n-1)$ spanning trees, because there will be $m-1$ possibilities if the edge from the sink vertex to v_1 is removed, likewise there will be $n-1$ possibilities if the edge from the sink vertex to u_{m-1} is removed. If both edges are removed, then clearly there will be $m+n-2$ spanning trees. Hence, the total number of spanning trees is $mn + 2(m-1)(n-1) + (m+n-2) = 3mn - m - n$. Therefore, $|\operatorname{Jac}(G)| = |\mathbb{Z}_m \times \mathbb{Z}_n| = 3mn - m - n$.

□

Corollary 2.1.2. *Let H be as in the theorem. If m and n are not relatively prime, then deleting the edge $\{u_{m-1}, v_1\}$ gives a graph whose Jacobian group has two nontrivial invariant factors. Thus, in this case, removing an edge causes the number of nontrivial invariant factors to increase.*

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