

New connections between the Abelian Sandpile Model and Domino Tilings

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# Abstract

This thesis examines the connections between the abelian sandpile model and the dimer model. Several original theorems are presented in the chapter investigating grid graphs with Klein Four group symmetry relating the number of symmetric recurrent configurations on grid graphs to the number of domino tilings on different checkerboards. A new proof for the number of tilings on a checkerboard is presented, as well as a partial new proof for the number of tilings on a Möbius checkerboard. The next chapter consists of a number of other theorems concerning specific graphs as well as recurrent configurations on grid graphs without symmetry. In the chapter exploring grid graphs with dihedral symmetry we find a relation between the number of symmetric configurations and weighted domino tilings on a class of graphs studied in [Pac97].



# Introduction

**“Who could ever calculate the path of a molecule? How do we know that the creations of worlds are not determined by falling grains of sand?” - Victor Hugo, Les Miserables**

Mandelbrot noticed that many naturally occurring systems can be thought of as fractals, which means that some correlation functions show non-trivial power law behavior. For example, the height profile of mountain ranges can be characterized by the dependence on  $R$  of the difference of height,  $\Delta h(R)$ , between two points separated by a distance  $R$ . It has been found that  $\Delta h(R)^2 \sim R^x$ . Similarly, river networks, earthquakes, brain activity, models of forest fires, biological evolution, and magnetism in metals can be described using power laws.

Bak, Tang and Wiesenfeld invented the idea of “self-organized criticality” in [BTW88] and argued that the dynamics which describe the power-law dependencies seen in non-equilibrium steady states in nature must not involve any fine-tuning of parameters. Such systems are driven to a state at the boundary between the stable and unstable states.

They also observed that if one builds a sandpile by dropping grains of sand on a flat circular table, one obtains a cone of sand. This system is invariably driven towards its critical states: it exhibits self-organized criticality.

The interest of this paper in the mathematics of the sandpile model and cellular automata thus arises from the physical phenomena mentioned. This project started with my adviser’s collaboration with Reed College graduate Daniela Morar which resulted in [MP08], continued with Harvey Mudd College graduate Natalie Durgin as presented in [Dur09], who has investigated symmetry in sandpiles, and further with his research with Reed College students Julia Porcino, Nick Salter and Tianyuan Xu.

This paper is mainly concerned with the connection between the recurrent states of the sandpile model and the dimer model. Even though this paper does not present the physical implications of the findings, there might be a connection between the dimer model and viral tiling theory, specifically the Caspar-Klug series, which asserts that viruses follow the structure of dominoes or polyominoes in the organization of their protein coats. This is presented in [TR06].

Chapter 1 presents the theory of the Abelian Sandpile Model and the chip-firing

game. Important concepts such as the Laplacian, the reduced Laplacian, and the sandpile group are introduced.

Chapter 2 describes the specifics of the model for symmetric recurrents. The actions of the symmetric recurrents are described through several propositions and corollaries. The important Matrix-Tree Theorem which is employed in several of our new theorems is introduced. Here we also introduce the theory of matchings, trees and dual trees. This process, which we refer to as *KPW* throughout this paper, due to Kenyon, Propp and Wilson [KPW], will be important for many of the constructive proofs.

Chapter 3 is the first original chapter. It consists of a series of theorems relating the number of Klein Four group - symmetric recurrents and tilings on grid graphs. Of interest is the new proof for Kasteleyn's formula for the number of domino tilings of an  $m \times n$  grid graph.

The next chapter presents a similar theorem for grid graphs with dihedral symmetry. An important class of graphs, which we refer to as *Ciucu graphs*, is also introduced.

The last chapter presents several new theorems for sandpile graphs without symmetry.

# Chapter 1

## The Abelian Sandpile Model

A reference for all the results in this section is [HLM].

This section will cover basic sandpile theory, setting up the stage for the next chapters.

Let  $\Gamma$  be a finite, weighted, directed graph with vertices  $V$  and edges  $E$ . The weight function is defined as

$$\text{wt}(v, w) = \text{the number of edges between } v \text{ and } w, \text{ where } v, w \in V.$$

Then, the outdegree and indegree for  $v \in V$  are defined as:

$$d_v = \text{outdeg}(v) = \sum_{w \in V} \text{wt}(v, w)$$
$$\text{indeg}(v) = \sum_{w \in V} \text{wt}(w, v)$$

Now we can define the sink of a graph: a vertex  $s \in V$  is a sink if  $d_s = 0$ . We call  $s$  a *global sink* if there is a directed path from each vertex to  $s$ . A global sink, if it exists, is unique. The graph  $\Gamma$  is undirected if  $\text{outdeg}(v) = \text{indeg}(v)$  for all  $v \in V$ .

**Definition 1.0.1** *A sandpile graph is a finite, directed graph with a global sink.*

Let  $X$  be a finite set. Then  $\mathbb{Z}X = \{ \sum_{x \in X} a_x x : a_x \in \mathbb{Z}, \forall x \in X \}$  is the free abelian group on  $X$ .

**Definition 1.0.2** *The Laplacian of  $\Gamma$  is the operator  $\Delta : \mathbb{Z}^V \rightarrow \mathbb{Z}^V$  defined by*

$$\Delta\phi(v) = \sum_{(w,v) \in E} (\phi(v) - \phi(w)) \text{ for } \phi \in \mathbb{Z}^V \text{ and } v \in V.$$

The *standard basis* for  $\mathbb{Z}^V$  is  $\{v^*\}_{v \in V}$  where

$$v^*(w) = \delta(v, w) = \begin{cases} 1 & \text{if } v = w \\ 0 & \text{if } v \neq w \end{cases}$$

We have

$$\Delta v^* = d_v v^* - \sum_{w \in V} \text{wt}(w, v) w^*.$$

We can get an isomorphism by fixing an ordering  $v_1, \dots, v_n$  on the vertices:

$$\begin{aligned} \mathbb{Z}^V &\xrightarrow{\sim} \mathbb{Z}^n \\ v_i^* &\rightarrow e_i \end{aligned}$$

where  $e_i$  is the  $i$ -th standard basis vector. Then,  $\Delta$  is a matrix given by:

$$\Delta_{ij} = \begin{cases} d_{v_i} - \text{wt}(v_i, v_i) & \text{if } i = j \\ -\text{wt}(v_i, v_j) & \text{if } i \neq j. \end{cases}$$

Now, take  $D = \text{diag}(d_{v_1}, \dots, d_{v_n})$  and the *adjacency matrix*,  $A$ , where  $A_{ij} = \text{wt}(v_i, v_j)$ . Now,

$$\Delta = D - A.$$

Let  $\tilde{V}$  denote the nonsink vertices of  $\Gamma$ . We can define two natural maps between  $\mathbb{Z}^V$  and  $\mathbb{Z}^{\tilde{V}}$ , a restriction map

$$\begin{aligned} \rho : \mathbb{Z}^V &\rightarrow \mathbb{Z}^{\tilde{V}} \\ \phi &\rightarrow \phi|_{\tilde{V}} \end{aligned}$$

and an extension map

$$i : \mathbb{Z}^{\tilde{V}} \rightarrow \mathbb{Z}^V$$

where

$$i(\phi)(v) = \begin{cases} \phi(v) & \text{if } v \in \tilde{V} \\ 0 & \text{otherwise.} \end{cases}$$

**Definition 1.0.3** *The reduced Laplacian of  $\Gamma$  is the operator  $\tilde{\Delta} : \mathbb{Z}^{\tilde{V}} \rightarrow \mathbb{Z}^{\tilde{V}}$  such that  $\tilde{\Delta} = \rho \circ \Delta \circ i$ .*

Thus, the matrix representing  $\tilde{\Delta}$  is obtained from the matrix representing  $\Delta$  by removing the rows and columns corresponding to sinks. A useful operation will be to take the transpose of the reduced Laplacian, the mapping

$$\tilde{\Delta}^t : \mathbb{Z}^{\tilde{V}} \rightarrow \mathbb{Z}^{\tilde{V}}$$

obtained by dualizing  $\tilde{\Delta}$ . Thus, for  $v \in \mathbb{Z}^{\tilde{V}}$ ,

$$\tilde{\Delta}^t v = d_v v - \sum_{w \in \tilde{V}} \text{wt}(v, w) w.$$

Restricting  $\Delta$  to  $\mathbb{Z}^{\tilde{V}}$  and setting  $s = 0$ , gives the *reduced Laplacian*,  $\tilde{\Delta} : \mathbb{Z}^{\tilde{V}} \rightarrow \mathbb{Z}^{\tilde{V}}$ . If  $v$  is an unstable vertex in a configuration  $c$ , firing  $v$  gives the new configuration

$$c - \tilde{\Delta} v.$$

There is a well-known isomorphism

$$\begin{aligned} \mathcal{S}(\Gamma) &\rightarrow \mathbb{Z}^{\tilde{V}} / \text{image}(\tilde{\Delta}) \\ c &\mapsto c. \end{aligned} \tag{1.1}$$



### 1.0.1 The Sandpile Group

**Definition 1.0.4** A sandpile or *configuration* on  $\Gamma$  is an element of  $\mathbb{N}\tilde{V}$ . A configuration  $c = \sum_{w \in \tilde{V}} c_w v$  is stable at a vertex  $v \in \tilde{V}$  if  $c_v < d_v$ . Otherwise it is unstable. A sandpile is stable if it is stable at each  $v \in \tilde{V}$ .

If  $c$  is unstable at  $v$ , we can *fire* or *topple*  $c$  at  $v$  to get a new configuration  $\tilde{c}$  defined by

$$\tilde{c} = \begin{cases} c_v - d_v + \text{wt}(v, v) & \text{if } w = v \\ c_w + \text{wt}(v, w) & \text{otherwise.} \end{cases}$$

for each  $w \in \tilde{V}$ . Thus,  $\tilde{c} = c - \tilde{\Delta}^t v$ . If we can get a configuration  $b$  starting from a configuration  $a$ , we write  $a \rightarrow b$ . Stabilization is independent of the order of firings, thus the name abelian sandpile model.

**Notation 1.0.5** If the configuration  $a$  has a stabilization, this is denoted  $a^\circ$ .

**Lemma 1.0.6** If  $\Gamma$  is a sandpile graph, then every configuration on  $\Gamma$  has a stabilization.

Note that the set of stable configurations on  $\Gamma$  is a commutative monoid under *stable addition*

$$(a + b) = (a + b)^\circ.$$

This means that the operation is addition in  $\mathbb{N}\tilde{V}$  followed by stabilization. The identity is the zero configuration.

**Definition 1.0.7** A configuration  $r$  is accessible if for each configuration  $s$ , there exists a configuration  $t$  such that  $t + s \rightarrow r$ . If  $r$  is stable, then  $r$  is *recurrent*.

Recurrent elements, which play an important role in this paper, should be defined in more detail.

**Definition 1.0.8** The maximal stable configuration on  $\Gamma$  is the configuration

$$c_{\max} = \sum_{v \in \tilde{V}} (d_v - 1)v$$

**Proposition 1.0.9** A configuration  $r$  is recurrent if and only if there exists a configuration  $s$  such that

$$r = (s + c_{\max})^\circ.$$

**Theorem 1.0.10** The collection of recurrent elements of  $\Gamma$  forms a group under stable addition, denoted  $\mathcal{S}(\Gamma)$  and called the sandpile group of  $\Gamma$ .

**Proposition 1.0.11** The following are equivalent:

- (1) the zero configuration is recurrent
- (2) every stable configuration is recurrent
- (3)  $\Gamma$  is acyclic.



# Chapter 2

## Symmetric sandpiles

A few of these results can also be found in [Dur09].

### 2.0.2 Sandpile Group Actions

Let  $G$  be a finite group. By an *action* of  $G$  on  $\Gamma$  with sink  $s$ , we mean a mapping

$$\begin{aligned} G \times V &\rightarrow V \\ (g, v) &\mapsto gv \end{aligned}$$

satisfying

1. if  $e$  is the identity of  $G$ , then  $ev = v$  for all  $v \in V$ ;
2.  $g(hv) = (gh)v$  for all  $g, h \in G$  and  $v \in V$ ;
3.  $gs = s$  for all  $g \in G$ ;
4. if  $(v, w) \in E$ , then  $(gv, gw) \in E$  and both edges have the same weight.

From now on, let  $G$  be a group acting on  $\Gamma$  with sink  $s$ .

By linearity, the action of  $G$  extends to an action on  $\mathbb{N}V$  and  $\mathbb{Z}V$ , and since  $G$  fixes the sink, it acts on configurations and each element of  $G$  induces a self-isomorphism of  $\mathcal{S}(\Gamma)$ .

**Notation 2.0.12** *Configuration  $c$  is symmetric (with respect to the action by  $G$ ) if  $gc = c$  for all  $g \in G$ .*

**Proposition 2.0.13** *The action of  $G$  commutes with stabilization. That is, if  $c$  is any configuration on  $\Gamma$ , then  $g(c^\circ) = (gc)^\circ$ .*

**Proof.** Stabilizing  $c$  consists of firing a sequence of vertices, say  $v_1, \dots, v_t$ . Then

$$c^\circ = c - \sum_{i=1}^t \tilde{\Delta}v_i.$$

At the  $k$ -th step in the stabilization process,  $c$  has relaxed to the configuration  $c' := c - \sum_{i=1}^k \tilde{\Delta}v_i$ . A vertex  $v$  is unstable in  $c'$  if and only if  $gv$  is unstable in  $gc' = gc - \sum_{i=1}^k \tilde{\Delta}(gv_i)$ . Thus, we can fire the sequence of vertices  $gv_1, \dots, gv_t$  in  $gc$ , resulting in the stable configuration

$$(gc)^\circ = gc - \sum_{i=1}^t \tilde{\Delta}(gv_i).$$

□

**Corollary 2.0.14** *The action of  $G$  preserves recurrent configurations, i.e., for each recurrent configuration  $c$  and each  $g \in \Gamma$ , it follows that  $gc$  is recurrent.*

**Proof.** If  $c$  is recurrent, we can find a configuration  $b$  such that  $c = (b + c_{\max})^\circ$ . Then,

$$\begin{aligned} gc &= g(b + c_{\max})^\circ \\ &= (gb + gc_{\max})^\circ \\ &= (gb + c_{\max})^\circ. \end{aligned}$$

Hence,  $gc$  is recurrent. □

**Corollary 2.0.15** *If  $c$  is a symmetric configuration, then so is its stabilization.*

**Proof.** For all  $g \in G$ , if  $gc = c$ , then  $gc^\circ = (gc)^\circ = c^\circ$ . □

**Proposition 2.0.16** *The collection of symmetric recurrent configurations forms a subgroup of the sandpile group,  $\mathcal{S}(\Gamma)$ .*

**Proof.** Since the group action respects addition in  $\tilde{N}\tilde{V}$  and stabilization, the sum of two symmetric recurrent configurations is again symmetric and recurrent. There is at least one symmetric recurrent configuration, namely,  $c_{\max}$ . Since the sandpile group is finite, it follows that these configurations form a subgroup. □

We will denote the subgroup of recurrent configurations of  $\mathcal{S}(\Gamma)$  by  $\mathcal{S}(\Gamma)^G$ .

For convenience, we now assume that the action of  $G$  is *faithful*, meaning that if  $g \in G$  and  $gv = v$  for all  $v \in V$ , then  $g$  is the identity. For each  $v \in V$ , the *orbit* of  $v$  under  $G$  is

$$Gv = \{gv : g \in G\}.$$

Let  $\mathcal{O} = \mathcal{O}(\Gamma, G) = \{Gv : v \in tV\}$  denote the set of orbits of the nonsink vertices of  $G$  under  $\Gamma$ . For  $v \in \tilde{V}$ , let  $[v] \in \mathcal{O}$  denote the corresponding orbit. Let

$$\tilde{\Delta}^G : \mathbb{Z}\mathcal{O} \rightarrow \mathbb{Z}\mathcal{O}$$

be the  $\mathbb{Z}$ -linear mapping defined on orbits  $[v]$  by

$$\tilde{\Delta}^G([v]) := \sum_{w \in [v]} \tilde{\Delta}(w).$$

**Proposition 2.0.17** *Let  $r: \mathbb{Z}\tilde{V}/\text{image}(\tilde{\Delta}) \rightarrow \mathcal{S}(\Gamma)$  denote the inverse of the isomorphism in (1.1). There is an isomorphism of groups defined by*

$$\begin{aligned} \phi: \mathbb{Z}\mathcal{O}/\text{image}(\tilde{\Delta}^G) &\rightarrow \mathcal{S}(\Gamma)^G \\ [v] &\mapsto r(\sum_{w \in [v]} w) \end{aligned}$$

for each  $v \in \tilde{V}$ .

**Proof.** Consider the homomorphism  $\mathbb{Z}\mathcal{O} \rightarrow \mathbb{Z}\tilde{V}/\text{image}(\tilde{\Delta})$  determined by  $[v] \mapsto \sum_{w \in [v]} w$  for  $v \in \tilde{V}$ . From the definition of  $\tilde{\Delta}^G$  it is clear that  $\text{image}(\tilde{\Delta}^G)$  is in the kernel, and thus there is an induced mapping,

$$\psi: \mathbb{Z}\mathcal{O}/\text{image}(\tilde{\Delta}^G) \rightarrow \mathbb{Z}\tilde{V}/\text{image}(\tilde{\Delta}).$$

Even though  $\psi([v])$  is  $G$ -symmetric, it is not immediately clear that the same can be said of  $r(\psi([v]))$ . To see this, consider the configuration  $|\mathcal{S}(\Gamma)|_{c_{\max}} \in \mathbb{Z}\tilde{V}$ . It is equivalent to  $\vec{0}$  modulo the image of  $\tilde{\Delta}$  given the isomorphism in (1.1). It is symmetric and can evidently be obtained by adding sand to  $c_{\max}$ . Therefore,

$$\phi([v]) = r(\psi([v])) = (|\mathcal{S}(\Gamma)|_{c_{\max}} + \psi([v]))^\circ$$

is symmetric.

It is clear that  $\phi$  is surjective, so it remains to show that  $\phi$  is injective. Suppose  $a = \tilde{\Delta}b$  and that  $a$  is symmetric. It suffices to show that  $b$  is symmetric. Fix  $g \in G$ , and consider the isomorphism  $g: \mathbb{Z}\tilde{V} \rightarrow \mathbb{Z}\tilde{V}$  determined by the action of  $G$  on vertices. We then have

$$\tilde{\Delta}b = a = ga = g\tilde{\Delta}b = (g\tilde{\Delta}g^{-1})(gb) = \tilde{\Delta}(gb).$$

It follows that  $b = gb$  for all  $g \in G$ , as required.  $\square$

**Corollary 2.0.18** *The number of recurrent configuration is*

$$|\mathcal{S}(\Gamma)^G| = \det \tilde{\Delta}^G.$$

### 2.0.3 Matchings and trees

In order to make the connection between the sandpile model and the dimer model on graphs, we need to introduce the concepts of spanning trees, dual tree, and an important process for this paper, which we will call *KPW*, after Kenyon, Propp and Wilson.

A *directed spanning tree* of  $\Gamma$  rooted at  $s$  is a directed subgraph containing all the vertices, having no directed cycles, and for which  $s$  has no out-going edges and every other vertex has exactly one outgoing edge. The weights of the edges of a directed spanning tree are the same as they are for  $\Gamma$ , and the weight of a spanning tree is the product of the weights of its edges.

**Theorem 2.0.19 (Matrix-Tree Theorem)** *The determinant of the reduced Laplacian,  $\tilde{\Delta}$ , defined in 1.0.3, is equal to the number of spanning trees.*

The proof of this can be found in many graph theory texts, so we omit it.

By the matrix-tree theorem, the sum of the weights of the set of all directed spanning trees of  $\Gamma$  rooted at  $s$  is equal to the determinant of  $\det \tilde{\Delta}$ . By using the fact that  $\mathcal{S}(\Gamma) \rightarrow \mathbb{Z}\tilde{V}/\text{image}(\tilde{\Delta})$ , and by taking the Smith normal form of  $\tilde{\Delta}$ , we conclude that the number of elements of the sandpile group is also the sum of the weights of the directed spanning trees rooted at  $s$ .

Now we can outline the generalized Temperley bijection, due to [KPW], between directed spanning trees of  $\Gamma$  rooted at  $s$  and perfect matchings of a related weighted undirected graph,  $\mathcal{H}(\Gamma)$ . Suppose that  $\Gamma$  is embedded in the plane. For each edge  $(u, v)$  of  $\Gamma$ , if  $(v, u)$  is also an edge, we assume  $\Gamma$  is embedded so that the two edges coincide. To create  $\mathcal{H}(\Gamma)$ , overlay the dual of  $\Gamma$  on top of  $\Gamma$  in the plane. The *dual tree* is constructed by drawing a vertex in the plane inside each face of  $\Gamma$ , including the unbounded face, then connecting these new vertices by edges if the corresponding faces of  $\Gamma$  which they represent are adjacent. Where these newly-added edges cross edges of  $\Gamma$ , add new vertices (these are drawn in white in Figure 2.1). Thus, every edge of  $\Gamma$  and its dual acquires a new vertex which we will call a *bisecting vertex*. Let  $\ell$  denote one of these bisecting vertices, and say it lies on the intersection of the edge  $(u, v)$  of  $\Gamma$  and the edge  $(f, g)$  of the dual graph. In  $\mathcal{H}$ , the edge  $(u, \ell)$  will now have weight  $\text{wt}(u, v)$ , and  $(\ell, v)$  weight  $\text{wt}(v, u)$ , and  $(f, \ell)$  and  $(g, \ell)$  weight 1. Processing each of the bisecting vertices in this way, we thus arrive at an undirected weighted graph. Remove any edges of weight 0 to arrive at the graph we will call  $\Gamma \cup \Gamma^*$ . Finally, pick any face,  $f_s$ , containing the sink vertex,  $s$  of  $\Gamma$  (In our applications,  $f_s$  will always turn out to be the unbounded face.) Remove the vertices of  $\Gamma \cup \Gamma^*$  corresponding to  $f_s$  and any incident edges of  $\Gamma \cup \Gamma^*$ . The remaining graph is  $\mathcal{H}(\Gamma)$ .

Figure 2.1 depicts a graph  $\Gamma$  with 6 vertices, embedded in the plane. Each edge denotes a pair of directed edges, with weights as indicated. However, the special weight of 0 denotes an absence of an edge in the indicated direction. Thus, from the vertex on the top left, there is an edge of weight 1 to a vertex directly below it, with no edge in the opposite direction.

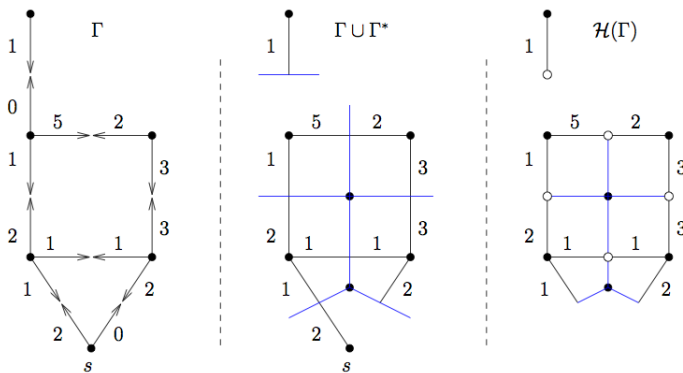


Figure 2.1: Construction of  $\mathcal{H}(\Gamma)$ .

For ease of display, we have omitted the vertex corresponding to the unbounded

face and have only partially drawn the edges connecting to that vertex in the picture of  $\Gamma \cup \Gamma^*$ .

A *perfect matching* of the graph  $\mathcal{H}(\Gamma)$  is a collection of its edges such that each vertex is incident with exactly one of the edges. The *weight* of a perfect matching is the product of the weights of its edges. We now describe the weight-preserving bijection between perfect matchings of  $\mathcal{H}(\Gamma)$  and directed spanning trees of  $\Gamma$  rooted at  $s$  due to [KPW]. Let  $T$  be a directed spanning tree of  $\Gamma$  rooted at  $s$ , and let  $\tilde{T}$  be the corresponding directed spanning tree of  $\Gamma^*$ , the dual of  $\Gamma$ , rooted at  $f_s$  (where  $f_s$  is the special face chosen in the construction of  $\mathcal{H}(\Gamma)$ ). For each vertex  $v$  of  $T$  except  $s$ , there is a unique out-going edge,  $e_v$ . In the construction of  $\mathcal{H}(\Gamma)$ , a bisecting vertex is added to the edge  $e_v$ . Call this vertex  $v'$ . In this way, we get an edge  $(v, v')$  of  $\mathcal{H}(\Gamma)$  for each nonsink vertex  $v \in T$ . Similarly, the vertices of  $\tilde{T}$  not equal to  $f_s$  give rise to edges of  $\mathcal{H}(\Gamma)$ . The collection of all edges arising this way from  $T$  and  $\tilde{T}$  gives the perfect matching of  $\mathcal{H}(\Gamma)$  corresponding to  $T$ .

The process of constructing  $\mathcal{H}(\Gamma)$  and the bijection between spanning trees of  $\Gamma$  and perfect matchings of  $\mathcal{H}(\Gamma)$  will be called *KPW*.

**Remark 2.0.20 (Perfect matchings and tilings)**

For the proofs and theorems following, we need to make the distinction between perfect matchings on grid graphs and domino tilings on checkerboards. An edge  $e$  in a perfect matching corresponds to a domino in a domino tiling covering the two squares whose centers are the vertices of  $e$ .

For example, consider the picture below:

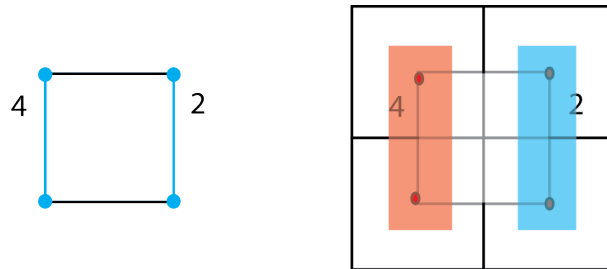


Figure 2.2: On the left a perfect matching of weight 8 of a  $2 \times 2$  grid, and on the right a tiling of weight 8 on the  $2 \times 2$  checkerboard.

We note that the edges of weights 4 and 2 respectively, constituting the perfect matching on the  $2 \times 2$  grid, correspond to the red and blue domino tilings on the squares shown.

**Remark 2.0.21 (Sandpile grids and regular grids)**

Throughout this thesis, it must be noted that when we refer to the grid, we mean the ordinary grid graph, whose perfect matchings we identify with domino tilings of a corresponding checkerboard. It must be not be confused with the sandpile grid, which we define below.

**Definition 2.0.22** *An  $m \times n$  sandpile grid is a grid graph with the edge elements connected to the sink and firing one grain of sand, except for the corner vertices which fire 2.*



# Chapter 3

## Grid graphs with Klein Four Group symmetry

**Definition 3.0.23** *The Klein Four group is the group  $\mathbb{Z}_2 \times \mathbb{Z}_2$ , the direct product of two copies of the cyclic group of order 2.*

First, we give an example of how the Klein Four Group symmetry looks on a grid graph.

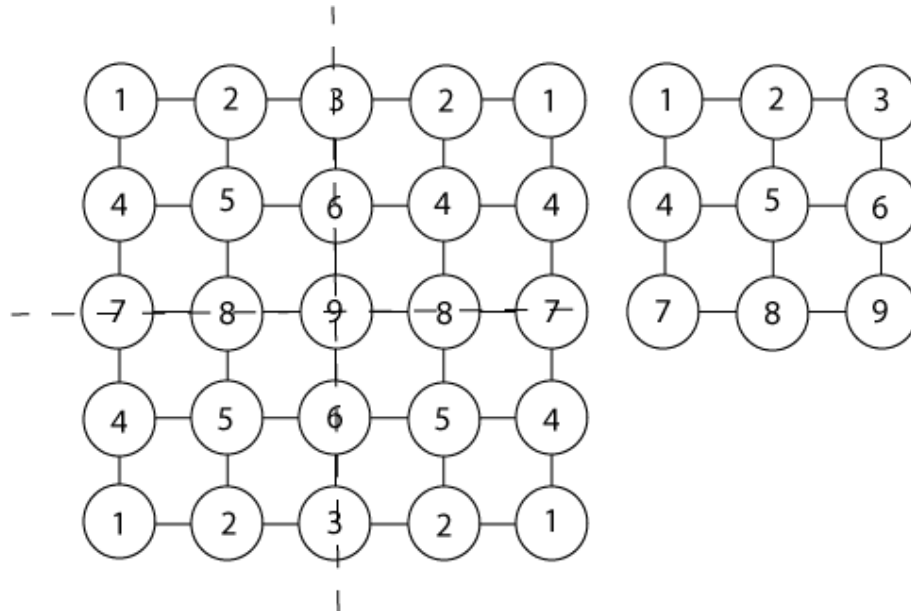


Figure 3.1: The graph on the right is invariant under the Klein Four group.

This section consists of a few theorems related to the dimer model, or domino tilings. These numbers can be found by expressing them as the Pfaffian of an antisymmetric matrix. This technique is applied for example in the classical, 2-dimensional computation of the dimer-dimer correlator function in statistical mechanics. It should be also noted that the number of tilings of a region is dependent on boundary conditions.

This section presents theorems relating the number of symmetric or non-symmetric recurrences to number of dimers on other structures, specifically Möbius strips or different grid graphs. A technique of computing the determinant of block tridiagonal matrices, presented in Appendix B, is used to give closed formulas in terms of Chebyshev polynomials, which are briefly described in Appendix A.

The theorem below shows that the domino tilings on an even by even grid graph are in bijection with the number of  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetric recurrences on the same graph. This theorem also presents a new proof of the number of domino tilings on an even by even grid graph through symmetric configurations in the abelian sandpile model.

**Theorem 3.0.24** *The following are equal:*

- (i) *The number of domino tilings of a  $2m \times 2n$  checkerboard,  $NG_{2m,2n}$ .*
- (ii) *the number of  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetric recurrences on a  $2m \times 2n$  sandpile grid,  $H_{2m,2n}$ .*
- (iii)  $\prod_{k=1}^n U_{2m} \left( i \cos \left( \frac{k\pi}{2n+1} \right) \right)$ .
- (iv)  $\prod_{h=1}^{2m} \prod_{k=1}^{2n} \left( 4 \cos^2 \frac{h\pi}{2m+1} + 4 \cos^2 \frac{k\pi}{2n+1} \right)$  from [Kas63].

**Proof.**

First, we know that (i) is equal to (ii) by the process we called *KPW*. Next, (ii) is equal to (iii) by computation of the determinant of the reduced Laplacian. Finally, (iii) and (iv) are equal by algebra.

First we show (ii) is equal to (iii). Let  $\Delta_{\text{sym}}$  be the Laplacian of the graph  $G^{\text{sym}}$  obtained by folding the  $2m \times 2n$  grid along its symmetries as described above. By Theorem 2.0.18,  $H_{2m,2n}$  can be identified with  $\mathcal{S}(G^{\text{sym}})$ , so by the Matrix-Tree Theorem it suffices to show the equality of  $\det \Delta_{\text{sym}}$  and  $NG_{2m,2n}$ . Recall that a tridiagonal matrix  $A = \{a_{i,j}\}$  has  $a_{i,j} = 0$  for  $|i-j| > 1$ , so that  $A$  has zero entries everywhere except possibly on the diagonal and immediately above and below. The structure of the grid graph gives  $\Delta_{\text{sym}}$  a block-tridiagonal form, relative to the ordering of the vertices that proceeds as one reads in English, from the top left corner across the rows to the bottom right. We first provide a formula for  $\det \Delta_{\text{sym}}$  by exploiting this structure. Let  $I_n$  be the  $n \times n$  identity matrix and  $A_n$  be the  $n \times n$  matrix with entries given by:

$$a_{i,j} = \begin{cases} 4 & \text{if } i = j \\ -1 & \text{if } |i-j| = 1, i \neq n \\ 3 & \text{if } i = n, j = n, \\ 0 & \text{if } |i-j| \geq 2 \end{cases}.$$

Relative to the aforementioned ordering of the vertices,  $\Delta_{\text{sym}}$  is a  $mn \times mn$  block-tridiagonal matrix of the form:

$$\Delta_{\text{sym}} = \begin{bmatrix} A_n & -I_n & & \cdots & & 0 \\ -I_n & A_n & -I_n & & & \\ & \ddots & \ddots & \ddots & & \vdots \\ & & -I_n & A_n & -I_n & \\ \vdots & & & \ddots & \ddots & \ddots \\ 0 & \cdots & & & -I_n & A_n & -I_n \\ & & & & & -I_n & B_n \end{bmatrix}$$

with  $B_n = A_n - I_n$ . Hence, by Theorem 2 in [Mol08],

$$\det \Delta_{\text{sym}} = (-1)^n \det T_{11} \quad (3.1)$$

where  $T_{11}$  is the upper-left block of size  $n \times n$  of the matrix

$$T = \begin{bmatrix} -B_n & I_n \\ I_n & 0 \end{bmatrix} \begin{bmatrix} A_n & -I_n \\ I_n & 0 \end{bmatrix}^{m-2} \begin{bmatrix} A_n & I_n \\ I_n & 0 \end{bmatrix}.$$

Set  $S_0 = I_n$ , and for all positive integers  $k$ , define

$$S_k = \left( \begin{bmatrix} A_n & -I_n \\ I_n & 0 \end{bmatrix}^{k-1} \begin{bmatrix} A_n & I_n \\ I_n & 0 \end{bmatrix} \right)_{11}$$

and

$$S'_k = \left( \begin{bmatrix} A_n & -I_n \\ I_n & 0 \end{bmatrix}^{k-1} \begin{bmatrix} A_n & I_n \\ I_n & 0 \end{bmatrix} \right)_{21},$$

where the subscript 21 denotes taking the lower left block of size  $n \times n$  of the parenthesized matrix. It follows that

$$S_0 = I_n, S_1 = A_n, \text{ and } S_k = A_n S_{k-1} - S_{k-2} \text{ for all } k \geq 2 \quad (3.2)$$

and

$$S'_k = S_{k-1} \text{ for all } k \geq 1.$$

By (3.2),  $S_k = U_k(\frac{1}{2}A_n)$ , where  $U_k(x)$  is the  $k$ -th Chebyshev polynomial of the second kind. This gives an expression for  $T_{11}$  in terms of Chebyshev polynomials:

$$\begin{aligned} T_{11} &= -B_n S_{m-1} + S'_{m-1} \\ &= -(A_n - I_n) S_{m-1} + S_{m-2} \\ &= S_{m-1} - (A_n S_{m-1} - S_{m-2}) \\ &= U_{m-1} \left( \frac{A_n}{2} \right) - U_m \left( \frac{A_n}{2} \right). \end{aligned}$$

Using the fact that the Chebyshev polynomials of the second kind satisfy

$$U_n(\cos \theta) = \frac{\sin((n+1)\theta)}{\sin \theta}$$

it is easy to check that the polynomial

$$p(x) := U_m\left(\frac{x}{2}\right) - U_{m-1}\left(\frac{x}{2}\right)$$

is a monic polynomial of degree  $m$  with zeros at  $x = 2 \cos \frac{(2k+1)\pi}{2m+1}$  for all integers  $0 \leq k \leq m-1$ . Thus,

$$T_{11} = -p(A) = - \prod_{k=0}^{m-1} \left( A_n - 2 \cos \frac{(2k+1)\pi}{2m+1} I_n \right).$$

Therefore, by (3.1), the determinant of  $\Delta_{\text{sym}}$  is given by

$$\begin{aligned} \det \Delta_{\text{sym}} &= (-1)^n \det \left( - \prod_{k=0}^{m-1} \left( A_n - 2 \cos \frac{(2k+1)\pi}{2m+1} I_n \right) \right) \\ &= \prod_{k=0}^{m-1} \det \left( A_n - 2 \cos \frac{(2k+1)\pi}{2m+1} I_n \right). \end{aligned}$$

Letting  $\chi_n$  denote the characteristic polynomial of  $A_n$  and setting  $t_{m,k} := 2 \cos \frac{(2k+1)\pi}{2m+1}$ , the above result rewrites as

$$\boxed{\det \Delta_{\text{sym}} = \prod_{k=0}^{m-1} \chi_n(t_{m,k})}. \quad (3.3)$$

We next show that the product expression in the above formula also counts the number of tilings of the  $2m \times 2n$  checkerboard, as obtained in [Kas63]. Let  $NG_{2m,2n}$  denote the number of domino tilings of the  $2m \times 2n$  checkerboard, given by:

$$NG_{2m,2n} = 2^{2mn} \prod_{j=1}^{2m} \prod_{k=1}^{2n} \left| \cos^2 \frac{j\pi}{2m+1} + \cos^2 \frac{k\pi}{2n+1} \right|$$

As shown in [ES10] Erica Shannon's Reed College thesis,

$$\begin{aligned} NG_{2m,2n} &= 2^{2mn} \prod_{j=1}^{2m} \prod_{k=1}^{2n} \left| \cos^2 \frac{j\pi}{2m+1} + \cos^2 \frac{k\pi}{2n+1} \right| \\ &= \left| \prod_{j=1}^{2m} \prod_{k=1}^{2n} \left( 2 \cos \frac{j\pi}{2m+1} + 2i \cos \frac{k\pi}{2n+1} \right) \right|^{1/2}. \end{aligned}$$

Then,

$$\left| \prod_{j=1}^{2m} \prod_{k=1}^{2n} \left( 2 \cos \frac{j\pi}{2m+1} + 2i \cos \frac{k\pi}{2n+1} \right) \right|^{1/2} = \left| \prod_{j=1}^{2m} \prod_{k=1}^{2n} \left( -2 \cos \frac{j\pi}{2m+1} - 2i \cos \frac{k\pi}{2n+1} \right) \right|^{1/2}$$

Define  $x_k$  and  $s_{2m,j}$  as follows:

$$x_k = 2i \cos \frac{k\pi}{2n+1}$$

$$s_{2m,j} = \prod_{j=1}^{2m} \left( x_k - 2 \cos \frac{j\pi}{2m+1} \right),$$

so that  $NG_{2m,2n} = \prod_{j=1}^{2m} s_{2m,j}$ . We notice  $\cos \frac{j\pi}{2m+1}$  is a root of the Chebyshev polynomial of the second kind, hence

$$s_{2m,j} = \prod_{j=1}^{2m} \left( x_k - 2 \cos \frac{j\pi}{2m+1} \right) = U_{2m} \left( \frac{x_k}{2} \right),$$

where  $U_{2m}(x)$  is the  $m$ -th Chebyshev polynomial of the second kind. To establish the equality of  $\det \Delta_{\text{sym}}$  and  $NG_{2m,2n}$ , we show by induction on  $n$  that for all  $0 \leq k \leq m-1$ ,

$$\chi_n(t_{m,k}) = s_{n,m-k}. \quad (3.4)$$

The base cases where  $n$  is equal to 1 and 2 can be easily verified. Recall the definition

$$\chi_n(x) = \det(A_n - xI_n).$$

Expanding the right-hand side of the above expression along the first row gives the recurrence

$$\chi_n(x) = (4 - x)\chi_{n-1}(x) - \chi_{n-2}(x) \quad \text{for all } n \geq 2.$$

Specializing to  $x = t_{m,k}$  then gives

$$\chi_n(t_{m,k}) = (4 - t_{m,k})\chi_{n-1}(t_{m,k}) - \chi_{n-2}(t_{m,k}) \quad \text{for all } n \geq 2.$$

Thus it suffices to show that  $s_{n,m-k} = U_{2m} \left( \frac{x_{m-k}}{2} \right)$  has this same recursion, i.e.,

$$U_{2m} \left( \frac{x_{m-k}}{2} \right) = (4 - t_{m,k})U_{2m-1} \left( \frac{x_{m-k}}{2} \right) - U_{2m-2} \left( \frac{x_{m-k}}{2} \right).$$

From the relation  $4 - t_{m,k} = x_{m-k}$ , the above equation follows immediately from the definition of the Chebyshev polynomials of the first kind via  $U_n(x) := 2xU_{n-1}(x) - U_{n-2}(x)$  for all  $n \geq 2$ .  $\square$

We write the following as a corollary as a quick reference for the interested reader.

**Corollary 3.0.25** *The number of domino tilings on any  $m \times n$  checkerboard is given by*

$$NG_{m,n} = \prod_{k=1}^{m/2} \prod_{j=1}^{n/2} \left( 4 \cos^2 \frac{k\pi}{m+1} + 4 \cos^2 \frac{j\pi}{n+1} \right) = \prod_{k=1}^{n/2} U_m \left( i \cos \left( \frac{k\pi}{n+1} \right) \right),$$

where  $i^2 = -1$ .

**Proof.** The proof follows directly from Theorem 3.0.24.  $\square$

After investigating the even by even grid, a natural question arises about the even by odd grid. We give its properties below, relating its number of  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetric recurrences to tilings of the even by even Möbius checkerboard and weighted tilings of the even by even grid. This constitutes a partial new proof of the number of domino tilings on the Möbius strip with the omission of the bijection between weighted domino tilings on the even by even grid and the number of domino tilings on an even by even Möbius checkerboard.

We show below an example of a tiling of the  $4 \times 4$  Möbius grid:

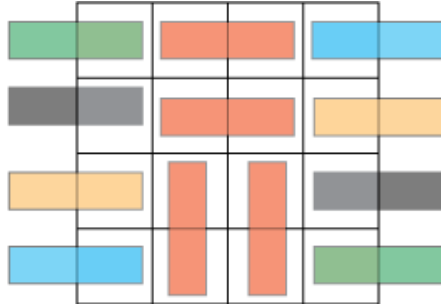


Figure 3.2: A tiling of the  $4 \times 4$  Möbius checkerboard.

**Theorem 3.0.26** *The following are equal:*

- (i) *The number of domino tilings of a  $2m \times 2n$  Möbius checkerboard,  $N_{2m,2n}$ .*
- (ii) *The number of  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetric recurrences on a  $2m \times (2n-1)$  sandpile grid,  $H_{2m,2n-1}$ .*
- (iii)  $2^m \prod_{k=1}^m T_{2n} \left( 1 + 2 \cos \left( \frac{k\pi}{2m+1} \right) \right)$ .
- (iv)  $\prod_{h=1}^m \prod_{k=1}^n \left( 4 \cos^2 \frac{h\pi}{2m+1} + 4 \sin^2 \frac{(4k-1)\pi}{4n} \right)$  from [LW03].
- (v) *the number of weighted domino tilings of a  $2m \times 2n$  checkerboard, as shown below:*



Figure 3.3: A  $9 \times 9$  checkerboard with tilings of weight 2, shown in red.

**Proof.** The proof is similar to the proof for the  $2m \times 2n$  sandpile grid with Klein Four Group symmetry.

First, (i) and (ii) are equal by KPW, (ii) and (iii) are equal by computation of the determinant of the reduced Laplacian matrix, (iii) and (iv) are equal by algebra and (ii) and (v) are again equal by KPW.

By Theorem 2.0.18,  $H_{2m,2n-1}$  can be identified with  $\mathcal{S}(G^{\text{sym}})$ , where  $G^{\text{sym}}$  is obtained by folding the  $2m \times (2n-1)$  grid along its symmetries. Then by the Matrix-Tree Theorem it suffices to show the equality of  $\det \Delta_{\text{sym}}$  and  $N_{2m,2n}$ . Recall that a tridiagonal matrix  $A = \{a_{i,j}\}$  has  $a_{i,j} = 0$  for  $|i-j| > 1$ , so that  $A$  has zero entries everywhere except possibly on the diagonal and immediately above and below. The structure of the grid graph gives  $\Delta_{\text{sym}}$  a block-tridiagonal form, relative to the ordering of the vertices that proceeds as one reads in English, from the top left corner across the rows to the bottom right. We first provide a formula for  $\det \Delta_{\text{sym}}$  by exploiting this structure. Let  $I_n$  be the  $n \times n$  identity matrix and  $A_n$  be the  $n \times n$  matrix with entries given by:

$$a_{i,j} = \begin{cases} 4 & \text{if } i = j \\ -1 & \text{if } |i-j| = 1, i \neq n \\ -2 & \text{if } i = n, j = n-1 \\ 0 & \text{if } |i-j| \geq 2 \end{cases}.$$

Relative to the aforementioned ordering of the vertices,  $\Delta_{\text{sym}}$  is a  $mn \times mn$  block-

tridiagonal matrix of the form:

$$\Delta_{\text{sym}} = \begin{bmatrix} A_n & -I_n & & \cdots & & 0 \\ -I_n & A_n & -I_n & & & \\ & \ddots & \ddots & \ddots & & \vdots \\ & & -I_n & A_n & -I_n & \\ \vdots & & & \ddots & \ddots & \ddots \\ 0 & & \cdots & & -I_n & A_n & -I_n \\ & & & & & -I_n & B_n \end{bmatrix}$$

with  $B_n = A_n - I_n$ . Hence, by Theorem 2 in [Mol08],

$$\det \Delta_{\text{sym}} = (-1)^n \det T_{11} \quad (3.5)$$

where  $T_{11}$  is the upper-left block of size  $n \times n$  of the matrix

$$T = \begin{bmatrix} -B_n & I_n \\ I_n & 0 \end{bmatrix} \begin{bmatrix} A_n & -I_n \\ I_n & 0 \end{bmatrix}^{m-2} \begin{bmatrix} A_n & I_n \\ I_n & 0 \end{bmatrix}.$$

Set  $S_0 = I_n$ , and for all positive integer  $k$ , define

$$S_k = \left( \begin{bmatrix} A_n & -I_n \\ I_n & 0 \end{bmatrix}^{k-1} \begin{bmatrix} A_n & I_n \\ I_n & 0 \end{bmatrix} \right)_{11}$$

and

$$S'_k = \left( \begin{bmatrix} A_n & -I_n \\ I_n & 0 \end{bmatrix}^{k-1} \begin{bmatrix} A_n & I_n \\ I_n & 0 \end{bmatrix} \right)_{21},$$

where the subscript 21 denotes taking the lower left block of size  $n \times n$  of the parenthesized matrix. It follows that

$$S_0 = I_n, S_1 = A_n, \text{ and } S_k = A_n S_{k-1} - S_{k-2} \text{ for all } k \geq 2 \quad (3.6)$$

and

$$S'_k = S_{k-1} \text{ for all } k \geq 1.$$

By (3.6),  $S_k = U_k(\frac{1}{2}A_n)$ , where  $U_k(x)$  is the  $k$ -th Chebyshev polynomial of the second kind. This gives an expression for  $T_{11}$  in terms of Chebyshev polynomials:

$$\begin{aligned} T_{11} &= -B_n S_{m-1} + S'_{m-1} \\ &= -(A_n - I_n) S_{m-1} + S_{m-2} \\ &= S_{m-1} - (A_n S_{m-1} - S_{m-2}) \\ &= U_{m-1} \left( \frac{A_n}{2} \right) - U_m \left( \frac{A_n}{2} \right). \end{aligned}$$

Using the fact that the Chebyshev polynomials of the second kind satisfy

$$U_n(\cos \theta) = \frac{\sin((n+1)\theta)}{\sin \theta}$$



it is easy to check that the polynomial

$$p(x) := U_m\left(\frac{x}{2}\right) - U_{m-1}\left(\frac{x}{2}\right)$$

is a monic polynomial of degree  $m$  with zeros at  $x = 2 \cos \frac{(2k+1)\pi}{2m+1}$  for all integers  $0 \leq k \leq m-1$ . This is different from the proof of the previous theorem, because our  $A_n$  matrices are not the same. Thus,

$$T_{11} = -p(A) = - \prod_{k=0}^{m-1} \left( A_n - 2 \cos \frac{(2k+1)\pi}{2m+1} I_n \right).$$

Therefore, by (3.5), the determinant of  $\Delta_{\text{sym}}$  is given by

$$\begin{aligned} \det \Delta_{\text{sym}} &= (-1)^n \det \left( - \prod_{k=0}^{m-1} \left( A_n - 2 \cos \frac{(2k+1)\pi}{2m+1} I_n \right) \right) \\ &= \prod_{k=0}^{m-1} \det \left( A_n - 2 \cos \frac{(2k+1)\pi}{2m+1} I_n \right). \end{aligned}$$

Letting  $\chi_n$  denote the characteristic polynomial of  $A_n$  and setting  $t_{m,k} := 2 \cos \frac{(2k+1)\pi}{2m+1}$ , the above result rewrites as

$$\boxed{\det \Delta_{\text{sym}} = \prod_{k=0}^{m-1} \chi_n(t_{m,k})}. \quad (3.7)$$

We next show that the product expression in the above formula also counts the number of tilings of the  $2m \times 2n$  Möbius strip, as obtained in [LW03]. Let  $N_{2m,2n}$  denote the number of domino tilings of the  $2m \times 2n$  Möbius strip, given by:

$$\begin{aligned} N_{2m,2n} &= \prod_{i=1}^m \prod_{j=1}^n \left( 4 \cos^2 \frac{i\pi}{2m+1} + 4 \sin^2 \frac{(4j-1)\pi}{4n} \right) \\ &= \prod_{i=1}^m \prod_{j=1}^n \left( \left( 4 \cos^2 \frac{i\pi}{2m+1} + 2 \right) - 2 \cos \frac{(4j-1)\pi}{2n} \right). \end{aligned}$$

Define  $x_i$  and  $s_{n,i}$  as follows:

$$\begin{aligned} x_i &= 4 \cos^2 \frac{i\pi}{2m+1} + 2 \\ s_{n,i} &= \prod_{j=1}^n \left( x_i - 2 \cos \frac{(4j-1)\pi}{2n} \right), \end{aligned}$$

so that  $N_{2m,2n} = \prod_{i=1}^m s_{n,i}$ . As  $j$  varies from 1 to  $n$ ,  $\cos \frac{(4j-1)\pi}{2n}$  and  $\cos \frac{(2j-1)\pi}{2n}$  take the same values with equal multiplicities, hence

$$s_{n,i} = \prod_{j=1}^n \left( x_i - 2 \cos \frac{(2j-1)\pi}{2n} \right) = 2T_n \left( \frac{x_i}{2} \right),$$

where  $T_n(x)$  is the  $n$ -th Chebyshev polynomial of the first kind. To establish the equality of  $\det \Delta_{\text{sym}}$  and  $N_{2m,2n}$ , we show by induction on  $n$  that for all  $0 \leq k \leq m-1$ ,

$$\chi_n(t_{m,k}) = s_{n,m-k}. \quad (3.8)$$

The base cases where  $n$  is equal to 1 and 2 can be easily verified. Recall the definition

$$\chi_n(x) = \det(A_n - xI_n).$$

Expanding the right-hand side of the above expression along the first row gives the recurrence

$$\chi_n(x) = (4 - x)\chi_{n-1}(x) - \chi_{n-2}(x) \quad \text{for all } n \geq 2.$$

Specializing to  $x = t_{m,k}$  then gives

$$\chi_n(t_{m,k}) = (4 - t_{m,k})\chi_{n-1}(t_{m,k}) - \chi_{n-2}(t_{m,k}) \quad \text{for all } n \geq 2.$$

Thus it suffices to show that  $s_{n,m-k} = 2T_n \left( \frac{x_{m-k}}{2} \right)$  has this same recursion, i.e.,

$$T_n \left( \frac{x_{m-k}}{2} \right) = (4 - t_{m,k})T_{n-1} \left( \frac{x_{m-k}}{2} \right) - T_{n-2} \left( \frac{x_{m-k}}{2} \right).$$

From the relation  $4 - t_{m,k} = x_{m-k}$ , the above equation follows immediately from the definition of the Chebyshev polynomials of the first kind via  $T_n(x) := 2xT_{n-1}(x) - T_{n-2}(x)$  for all  $n \geq 2$ . Everything else is similar to the proof in Theorem 3.0.24, so we can associate (ii) and (iii).

In order to prove that the number of symmetric recurrents on an even by odd sandpile grid graph is also the number of weighted tilings on an even by even checkerboard, we look at the reduced Laplacian, which encodes the firings of another graph. Applying KPW to this graph gives the connection with the weighted checkerboard.  $\square$

# Chapter 4

## Grid graphs with dihedral symmetry

In this section, we explore the abelian sandpile model on the sandpile grid graph with dihedral symmetry. The equivalence classes for the  $5 \times 5$  sandpile grid graph are shown in the picture below:

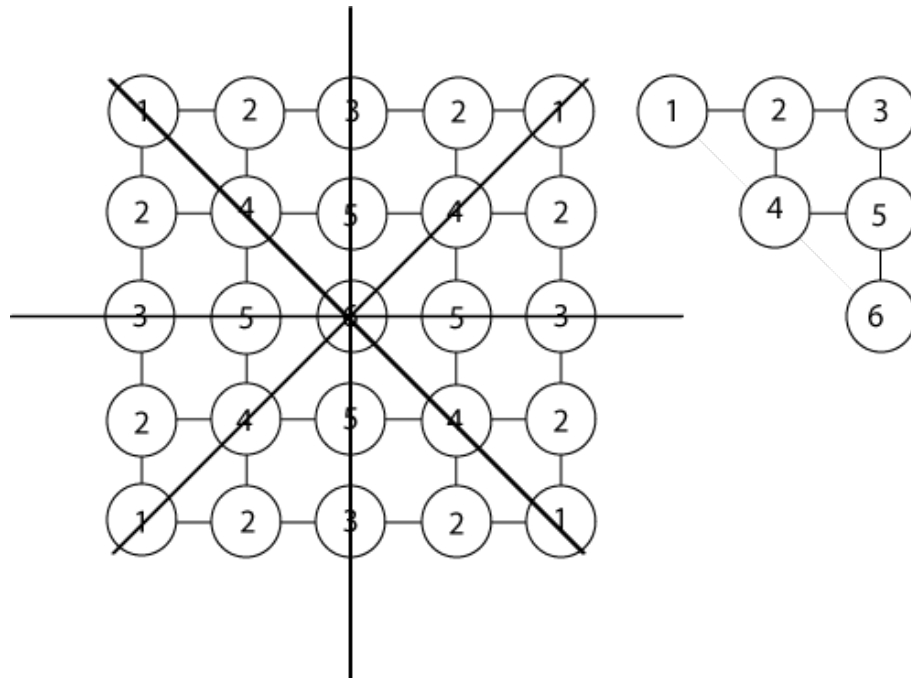


Figure 4.1: The graph on the right is invariant under the dihedral group  $D_4$ .

We will show that the number of recurrents on this kind of graph equals the number of perfect matchings of the Ciucu graphs  $H_n$ , as defined in [Pac97].

An interesting result involving these graphs is Lemma 2 in [Pac97].

**Theorem 4.0.27** *Let  $NH_n$  be the number of domino tilings of  $H_n$ . The number of domino tilings of the square checkerboard is given by*

$$N(2n, 2n) = 2^n (NH_n)^2.$$

The combinatorial proof of the theorem can be found in the same paper.

The graphs  $H_n$  are presented below:

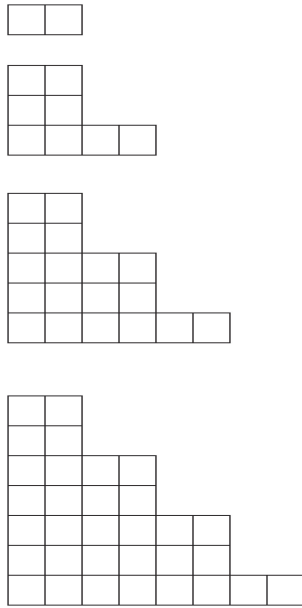


Figure 4.2: The Ciucu graphs  $H_1, H_2, H_3, H_4$ .

Let  $P_n$  be defined as below:

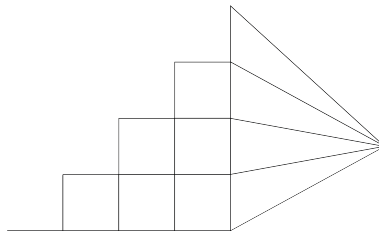


Figure 4.3: The graph  $P_n$ .

A result concerning the numbers  $a_n$  is given in [MP08]:

**Theorem 4.0.28** *The order of the sandpile group of the graph  $P_n$  is  $a_n$ .*

**Proof.** The result is immediate by applying KPW and then appealing to [Pac97].  $\square$   
 Now, to formulate the main theorem of this chapter:

**Theorem 4.0.29** *Let  $NH_{n-1}$  denote the weighted tilings on the Ciucu graph  $H_{n-1}$  as shown below for the cases when  $n$  is odd or even. Let  $NG_{2m,2n}$  denote the number of recurrences of the  $2m \times 2n$  sandpile grid graph with dihedral symmetry. Then,  $NH_{n-1} = NG_{2m,2n}$ .*

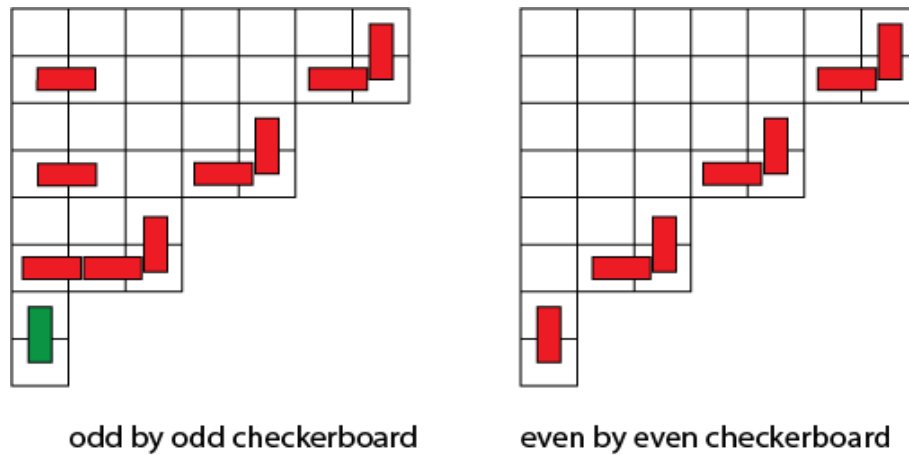


Figure 4.4: The  $H_4$  checkerboard with tilings of weight 2, shown in red, corresponding to the  $9 \times 9$  and  $10 \times 10$  grid graph with dihedral symmetry.

**Proof.**

We can arrive at this result by performing the same operations as described for the other proofs of this paper.

By first computing the Laplacian of the sandpile grid graph with dihedral symmetry, we get a special tridiagonal block matrix, with blocks of decreasing size. This Laplacian matrix is of the following form

$$\begin{bmatrix} A_{n-5} & -I_{n-6} & & \cdots & & 0 \\ -I'_{n-6} & A_{n-6} & -I_{n-7} & & & \\ & \ddots & \ddots & \ddots & & \vdots \\ & & -I'_k & A_k & -I_{k-1} & \\ \vdots & & & \ddots & \ddots & \ddots \\ 0 & & \cdots & & -I'_2 & A_2 & -I_1 \\ & & & & & -I'_1 & 2 \end{bmatrix},$$

where the  $A_n$  matrices are of the following form:

$$\begin{bmatrix} 4 & -2 & & \cdots & & 0 \\ -1 & 4 & -1 & & & \\ & \ddots & \ddots & \ddots & & \vdots \\ & & -1 & 4 & -1 & \\ \vdots & & & \ddots & \ddots & \ddots \\ 0 & \cdots & & -1 & 4 & -1 \\ & & & & -1 & 3 \end{bmatrix},$$

and the  $I'_n$  matrices are the identity matrices with a -2 in the position corresponding to the first row and the first column.

Next, we create the graph  $\Gamma$  from the transpose of the Laplacian matrix. This graph, and its dual are shown below:

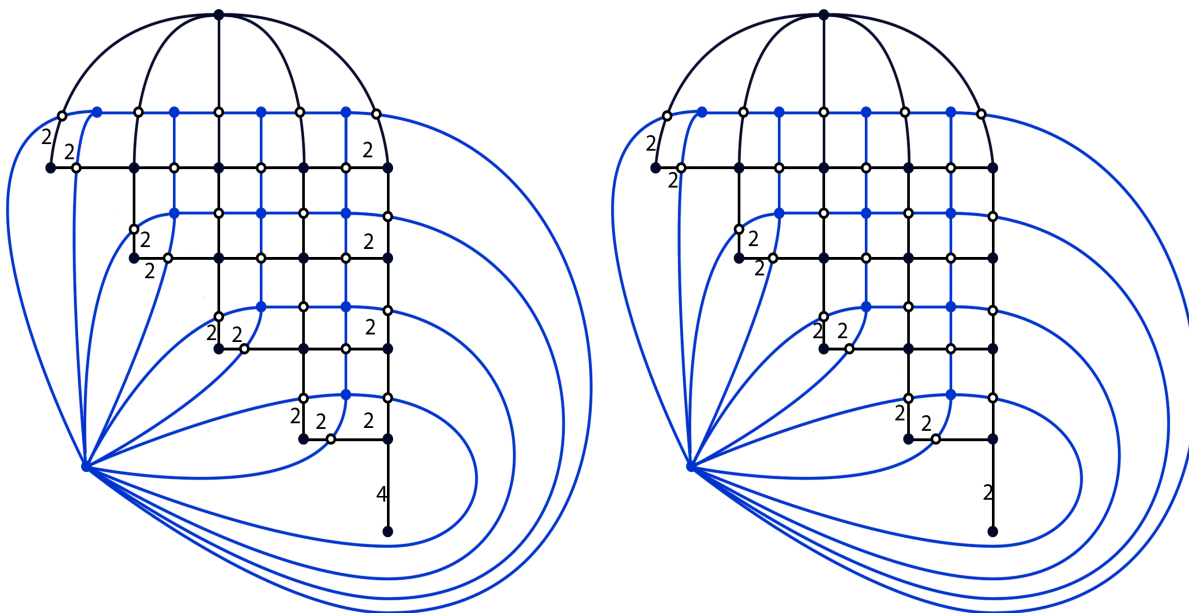


Figure 4.5: The resulting graph  $\Gamma$  and its dual for a  $9 \times 9$  or  $10 \times 10$  grid graph.

Overlaying the dual on top of the graph, and removing the edges of weight 0 from the sink, which correspond to all the curved black and blue, lines we arrive at the Ciucu graphs  $H_n$  as shown in Figure 5.2.

Now, by KPW, we can associate to each symmetric recurrent a weighted domino tiling as in the statement of the proof.  $\square$

# Chapter 5

## Other connections between the sandpile model and the dimer model

The theorems presented in this chapter are similar in content to the ones presented before. The difference is that they do not involve sandpile grid graphs with symmetry.

**Theorem 5.0.30** *The number of domino tilings on a  $3 \times 2n$  checkerboard is the number of domino tilings on a  $2n \times 2$  Möbius strip, for  $n \geq 1$ .*

**Proof.**

Let  $a_n$  denote the number of ways to tile a  $3 \times 2n$  grid of squares using domino tilings. Examining cases, we derive the recurrence:

$$a_n = 3a_{n-1} + 2a_{n-2} + 2a_{n-3} + \cdots + 2a_1 + 2a_0.$$

Then we note that

$$a_n - a_{n-1} = 3a_{n-1} - a_{n-2}$$

So then,

$$a_n = 4a_{n-1} - a_{n-2}$$

for  $n \geq 2$ , with initial conditions  $a_0 = 1$  and  $a_1 = 3$ .

We can prove in a similar fashion that  $A(2n)$ , the number of tilings on a  $2n \times 2$  Möbius strip, satisfies the following recursion:  $A(2n) = 4A(2n-1) - A(2n-2)$ . Then the result follows from the fact that the two sequences satisfy the same recursion.  $\square$

**Theorem 5.0.31** *The number of recurrent configurations on the  $m \times n$  sandpile grid graph with one vertex designated as the sink is the number of spanning trees on the  $m \times n$  grid graph. These are also counted by*

$$D_{m,n} = \prod_{h=1}^{m-1} \prod_{k=1}^{n-1} \left( 4 \sin^2 \frac{h\pi}{2m} + 4 \sin^2 \frac{k\pi}{2n} \right) = \prod_{h=1}^{m-1} U_{n-1} \left( 2 - \cos \frac{h\pi}{m} \right),$$

where we know the first equality by [KG78].

**Proof.**

Using the fact that  $\cos \frac{k\pi}{n}$  is a root of the second kind of Chebyshev polynomials, and the fact that  $\cos(2\theta) = 1 - 2\sin^2(\theta)$ ,

$$\begin{aligned} D_{m,n} &= \prod_{h=1}^{m-1} \prod_{k=1}^{n-1} \left( 4 \sin^2 \frac{h\pi}{2m} + 4 \sin^2 \frac{k\pi}{2n} \right) \\ &= \prod_{h=1}^{m-1} \prod_{k=1}^{n-1} \left( 4 - 2 \cos \frac{h\pi}{m} - 2 \cos \frac{k\pi}{n} \right) \\ &= \prod_{h=1}^{m-1} U_{n-1} \left( 2 - \cos \frac{h\pi}{m} \right). \end{aligned}$$

□

Next, we further investigate the number of recurrents on any grid graph and relate it to the tilings on the even by odd grid graph again.

**Theorem 5.0.32** *Let  $H_{2m,2n+1}$  denote the tilings on a  $2m \times (2n + 1)$  checkerboard. Let  $NG_{m,n}$  denote the number of recurrents of the  $m \times n$  sandpile grid graph with the right corner elements modified to fire 1 instead of 2 to the sink and the right edge elements, except for the corner elements, modified to fire 0 instead of 1 to the sink.*

*Then,  $|H_{2m,2n+1}| = NG_{m,n}$ . In addition, these are counted by the following Chebyshev polynomial formula:  $\prod_{k=1}^n U_{2m} \left( i \cos \left( \frac{k\pi}{2n+2} \right) \right)$ .*

**Proof.**

The reduced Laplacian matrix of this grid graph with special firings is

$$\Delta = \begin{bmatrix} A_n & -I_n & & \cdots & & 0 \\ -I_n & A_n & -I_n & & & \\ & \ddots & \ddots & \ddots & & \vdots \\ & & -I_n & A_n & -I_n & \\ \vdots & & & \ddots & \ddots & \ddots \\ 0 & \cdots & & & -I_n & A_n & -I_n \\ & & & & & -I_n & A_n \end{bmatrix}$$

where  $I_n$  is the  $n \times n$  identity matrix and  $A_n$  is the  $n \times n$  matrix with entries given by

$$a_{i,j} = \begin{cases} 4 & \text{if } i = j \\ -1 & \text{if } |i - j| = 1, i \neq n \\ 3 & \text{if } i = n, j = n, \\ 0 & \text{if } |i - j| \geq 2 \end{cases}.$$



Then the determinant is computed by similar methods as in the other theorems. By theorem 2 in [Mol08],

$$\det \Delta = (-1)^n \det T_{11} \quad (5.1)$$

where  $T_{11}$  is the upper-left block of size  $n \times n$  of the matrix

$$T = \begin{bmatrix} -A_n & I_n \\ I_n & 0 \end{bmatrix} \begin{bmatrix} A_n & -I_n \\ I_n & 0 \end{bmatrix}^{m-2} \begin{bmatrix} A_n & I_n \\ I_n & 0 \end{bmatrix}.$$

Set  $S_0 = I_n$ , and for all positive integer  $k$ , define

$$S_k = \left( \begin{bmatrix} A_n & -I_n \\ I_n & 0 \end{bmatrix}^{k-1} \begin{bmatrix} A_n & I_n \\ I_n & 0 \end{bmatrix} \right)_{11}$$

and

$$S'_k = \left( \begin{bmatrix} A_n & -I_n \\ I_n & 0 \end{bmatrix}^{k-1} \begin{bmatrix} A_n & I_n \\ I_n & 0 \end{bmatrix} \right)_{21},$$

where the subscript 21 denotes taking the lower left block of size  $n \times n$  of the parenthesized matrix. It follows that

$$S_0 = I_n, S_1 = A_n, \text{ and } S_k = A_n S_{k-1} - S_{k-2} \text{ for all } k \geq 2 \quad (5.2)$$

and

$$S'_k = S_{k-1} \text{ for all } k \geq 1.$$

By (5.2),  $S_k = U_k(\frac{1}{2}A_n)$ , where  $U_k(x)$  is the  $k$ -th Chebyshev polynomial of the second kind. This gives an expression for  $T_{11}$  in terms of Chebyshev polynomials:

$$\begin{aligned} T_{11} &= -A_n S_{m-1} + S'_{m-1} \\ &= -A_n S_{m-1} + S_{m-2} \\ &= U_m \left( \frac{A_n}{2} \right). \end{aligned}$$

The polynomial  $p(x) = U_m(\frac{x}{2})$  has zeros at  $x = 2 \cos \frac{k\pi}{n+1}$  for all integers  $0 \leq k \leq m-1$ . Thus,

$$T_{11} = -p(A) = - \prod_{k=0}^{m-1} \left( A_n - 2 \cos \frac{k\pi}{n+1} I_n \right).$$

Therefore, by (5.1), the determinant of  $\Delta$  is given by

$$\begin{aligned} \det \Delta &= (-1)^n \det \left( - \prod_{k=0}^{m-1} \left( A_n - 2 \cos \frac{k\pi}{n+1} I_n \right) \right) \\ &= \prod_{k=0}^{m-1} \det \left( A_n - 2 \cos \frac{k\pi}{n+1} I_n \right). \end{aligned}$$

Letting  $\chi_n$  denote the characteristic polynomial of  $A_n$  and setting  $t_{m,k} := 2 \cos \frac{k\pi}{n+1}$ , the above result rewrites as

$$\boxed{\det \Delta = \prod_{k=0}^{m-1} \chi_n(t_{m,k})}. \quad (5.3)$$

We next show that the product expression in the above formula also counts the number of tilings of the  $2m \times (2n+1)$  checkerboard. Let  $N_{2m,2n+1}$  denote the number of domino tilings of the  $2m \times (2n+1)$  checkerboard, given by:

$$N_{2m,2n+1} = 2^{2mn} \prod_{j=1}^{2m} \prod_{k=1}^{2n+1} \left| \cos^2 \frac{j\pi}{2m+1} + \cos^2 \frac{k\pi}{2n+2} \right|.$$

As shown in [ES10],

$$\begin{aligned} N_{2m,2n+1} &= 2^{2mn} \prod_{j=1}^{2m} \prod_{k=1}^{2n+1} \left| \cos^2 \frac{j\pi}{2m+1} + \cos^2 \frac{k\pi}{2n+2} \right| \\ &= \left| \prod_{j=1}^{2m} \prod_{k=1}^{2n+1} \left( 2 \cos \frac{j\pi}{2m+1} + 2i \cos \frac{k\pi}{2n+2} \right) \right|^{1/2}. \end{aligned}$$

Then,

$$\begin{aligned} &\left| \prod_{j=1}^{2m} \prod_{k=1}^{2n+1} \left( 2 \cos \frac{j\pi}{2m+1} + 2i \cos \frac{k\pi}{2n+2} \right) \right|^{1/2} \\ &= \left| \prod_{j=1}^{2m} \prod_{k=1}^{2n+1} \left( -2 \cos \frac{j\pi}{2m+1} - 2i \cos \frac{k\pi}{2n+2} \right) \right|^{1/2} \end{aligned}$$

Define  $x_k$  and  $s_{2m,j}$  as follows:

$$\begin{aligned} x_k &= 2i \cos \frac{k\pi}{2n+2} \\ s_{2m,j} &= \prod_{k=1}^{2n+1} \left( x_k - 2 \cos \frac{j\pi}{2m+1} \right), \end{aligned}$$

so that  $N_{2m,2n+1} = \prod_{j=1}^{2m} s_{2m,j}$ . We notice  $\cos \frac{j\pi}{2m+1}$  is a root of the Chebyshev polynomial of the second kind, hence

$$s_{2m,j} = \prod_{k=1}^{2n+1} \left( x_k - 2 \cos \frac{j\pi}{2m+1} \right) = U_{2m} \left( \frac{x_k}{2} \right),$$

where  $U_{2m}(x)$  is the  $2m$ -th Chebyshev polynomial of the second kind. To establish the equality of  $\det \Delta$  and  $N_{2m,2n+1}$ , we show by induction on  $n$  that for all  $0 \leq k \leq m-1$ ,

$$\chi_n(t_{m,k}) = s_{n,m-k}. \quad (5.4)$$

The base cases where  $n$  is equal to 1 and 2 can be easily verified. Recall the definition

$$\chi_n(x) = \det(A_n - xI_n).$$

Expanding the right-hand side of the above expression along the first row gives the recurrence

$$\chi_n(x) = (4 - x)\chi_{n-1}(x) - \chi_{n-2}(x) \quad \text{for all } n \geq 2.$$

Specializing to  $x = t_{m,k}$  then gives

$$\chi_n(t_{m,k}) = (4 - t_{m,k})\chi_{n-1}(t_{m,k}) - \chi_{n-2}(t_{m,k}) \quad \text{for all } n \geq 2.$$

Thus it suffices to show that  $s_{n,m-k} = U_{2m}\left(\frac{x_{m-k}}{2}\right)$  has this same recursion, i.e.,

$$U_{2m}\left(\frac{x_{m-k}}{2}\right) = (4 - t_{m,k})U_{2m-1}\left(\frac{x_{m-k}}{2}\right) - U_{2m-2}\left(\frac{x_{m-k}}{2}\right).$$

From the relation  $4 - t_{m,k} = x_{m-k}$ , the above equation follows immediately from the definition of the Chebyshev polynomials of the first kind via  $U_n(x) := 2xU_{n-1}(x) - U_{n-2}(x)$  for all  $n \geq 2$ .  $\square$

Next, we find another relation between the recurrences on any grid graph and domino tilings on a bigger graph with an internal cell removed. By an internal cell removed, we mean one that is not a corner cell.

**Theorem 5.0.33** *The following numbers are equal, for  $m$  and  $n$  both  $\geq 2$ :*

- (i) *the number of recurrences on an  $m \times n$  sandpile grid graph = the number of spanning trees on the  $m \times n$  sandpile grid graph*
- (ii) *the number of spanning trees on the  $(m+1) \times (n+1)$  grid graph*
- (iii) *the number of domino tilings on a  $(2m+1) \times (2n+1)$  checkerboard with an internal edge cell removed*
- (iv) *the number of domino tilings on a  $(2m+1) \times (2n+1)$  checkerboard with the left corner cell removed (from [OEIS]).*

When  $n=2$ , we get the number of domino tilings on the  $2m \times 5$  checkerboard with weighted tilings on the lower left and right corner tilings, as shown below, while when  $m=2$  we get the number of domino tilings on the  $5 \times (2n+1)$  checkerboard with the three last cells in the last row and a rightmost weighted tiling on the row before last removed as shown in Figure 5.2:

$$\text{Furthermore, these numbers are equal to } \prod_{h=1}^{m-1} \prod_{k=1}^{n-1} \left( 4 \sin^2 \frac{h\pi}{2m} + 4 \sin^2 \frac{k\pi}{2n} \right).$$

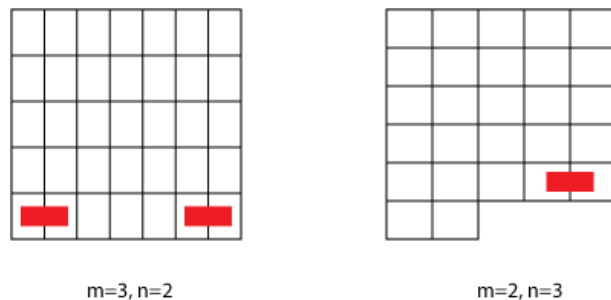


Figure 5.1: On the left example when  $n=2$  and on the right example when  $m=2$ .

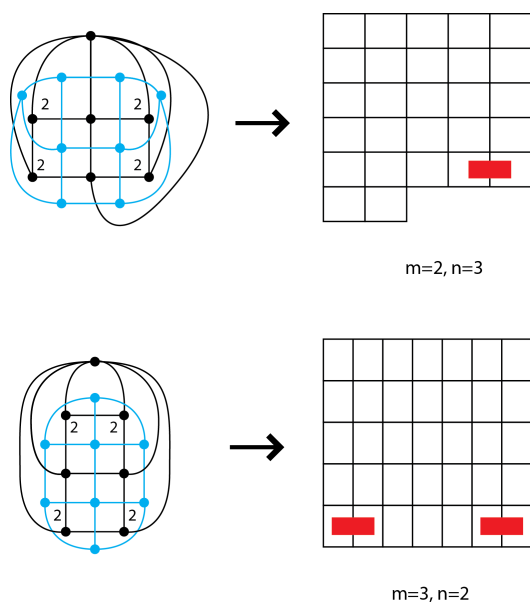


Figure 5.2: Applying KPW for our two base cases.

**Proof.** First, let's deal with the base cases. We give the constructive, visual proof for the cases first when  $m = 3$ ,  $n = 2$  and second when  $m = 2$ ,  $n = 3$  in Figure 5.1. This generalizes for when  $m$  varies in the first case and then  $n$  varies in the second case.

We will first show that (i) and (ii) are equal by proving that the determinant of the reduced Laplacian for the  $m \times n$  sandpile grid graph is equal to the formula above for the number of spanning trees on the  $(m + 1) \times (n + 1)$  grid graph.

We will use the same argument as in the proofs above referencing Molinari's proof in [Mol08]. We first provide a formula for  $\Delta$  by exploiting this structure. Let  $I_n$  be

the  $n \times n$  identity matrix and  $A_n$  be the  $n \times n$  matrix with entries given by:

$$a_{i,j} = \begin{cases} 4 & \text{if } i = j \\ -1 & \text{if } |i - j| = 1, i \neq n. \\ 0 & \text{if } |i - j| \geq 2 \end{cases}$$

Relative to the aforementioned ordering of the vertices,  $\Delta$  is a  $mn \times mn$  block-tridiagonal matrix of the form:

$$\Delta = \begin{bmatrix} A_n & -I_n & & \cdots & & 0 \\ -I_n & A_n & -I_n & & & \\ & \ddots & \ddots & \ddots & & \vdots \\ & & -I_n & A_n & -I_n & \\ \vdots & & & \ddots & \ddots & \ddots \\ 0 & \cdots & & -I_n & A_n & -I_n \\ & & & & -I_n & A_n \end{bmatrix}.$$

Hence, by Theorem 2 in [Mol08],

$$\det \Delta = (-1)^n \det T_{11} \quad (5.5)$$

where  $T_{11}$  is the upper-left block of size  $n \times n$  of the matrix

$$T = \begin{bmatrix} -A_n & I_n \\ I_n & 0 \end{bmatrix} \begin{bmatrix} A_n & -I_n \\ I_n & 0 \end{bmatrix}^{m-2} \begin{bmatrix} A_n & I_n \\ I_n & 0 \end{bmatrix}.$$

Set  $S_0 = I_n$ , and for all positive integer  $k$ , define

$$S_k = \left( \begin{bmatrix} A_n & -I_n \\ I_n & 0 \end{bmatrix}^{k-1} \begin{bmatrix} A_n & I_n \\ I_n & 0 \end{bmatrix} \right)_{11}$$

and

$$S'_k = \left( \begin{bmatrix} A_n & -I_n \\ I_n & 0 \end{bmatrix}^{k-1} \begin{bmatrix} A_n & I_n \\ I_n & 0 \end{bmatrix} \right)_{21},$$

where the subscript 21 denotes taking the lower left block of size  $n \times n$  of the parenthesized matrix. It follows that

$$S_0 = I_n, S_1 = A_n, \text{ and } S_k = A_n S_{k-1} - S_{k-2} \text{ for all } k \geq 2 \quad (5.6)$$

and

$$S'_k = S_{k-1} \text{ for all } k \geq 1.$$

By (5.6),  $S_k = U_k(\frac{1}{2}A_n)$ , where  $U_k(x)$  is the  $k$ -th Chebyshev polynomial of the second kind. This gives an expression for  $T_{11}$  in terms of Chebyshev polynomials:

$$\begin{aligned} T_{11} &= -A_n S_{m-1} + S'_{m-1} \\ &= -A_n S_{m-1} + S_{m-2} \\ &= U_{m-1} \left( \frac{A_n}{2} \right). \end{aligned}$$

Using the fact that the Chebyshev polynomials of the second kind satisfy

$$U_n(\cos \theta) = \frac{\sin((n+1)\theta)}{\sin \theta}$$

it is easy to check that the polynomial

$$p(x) := U_m \left( \frac{x}{2} \right)$$

is a monic polynomial of degree  $m$  with zeros at  $x = 2 \cos \frac{k\pi}{m+1}$  for all integers  $0 \leq k \leq m-1$ . Thus,

$$T_{11} = -p(A) = - \prod_{k=0}^{m-1} \left( A_n - 2 \cos \frac{k\pi}{m+1} I_n \right).$$

Therefore, by (5.5), the determinant of  $\Delta$  is given by

$$\begin{aligned} \det \Delta &= (-1)^n \det \left( - \prod_{k=0}^m \left( A_n - 2 \cos \frac{k\pi}{m+1} I_n \right) \right) \\ &= \prod_{k=0}^{m-1} \det \left( A_n - 2 \cos \frac{k\pi}{m+1} I_n \right). \end{aligned}$$

Letting  $\chi_n$  denote the characteristic polynomial of  $A_n$  and setting  $t_{m,k} := 2 \cos \frac{k\pi}{m+1}$ , the above result rewrites as

$$\boxed{\det \Delta = \prod_{k=0}^{m-1} \chi_n(t_{m,k})}. \quad (5.7)$$

We already know that the formula for the number of spanning trees is

$$\prod_{h=1}^{m-1} \prod_{k=1}^{n-1} \left( 4 \sin^2 \frac{h\pi}{2m} + 4 \sin^2 \frac{k\pi}{2n} \right) = \prod_{h=1}^{m-1} U_{n-1} \left( 2 - \cos \frac{h\pi}{m} \right)$$

from Theorem 5.0.31.

To establish the equality of  $\det \Delta$  and the number of spanning trees on the sandpile grid graph, we show by induction on  $n$  that for all  $0 \leq k \leq m - 1$ ,

$$\chi_n(t_{m,k}) = s_{n,m-k}, \quad (5.8)$$

where  $s_{n,m-k} = U_{n-1}\left(\frac{x_{m-k}}{2}\right)$ .

The base cases where  $n$  is equal to 1 and 2 can be easily verified. Recall the definition

$$\chi_n(x) = \det(A_n - xI_n).$$

Expanding the right-hand side of the above expression along the first row gives the recurrence

$$\chi_n(x) = (4 - x)\chi_{n-1}(x) - \chi_{n-2}(x) \quad \text{for all } n \geq 2.$$

Specializing to  $x = t_{m,k}$  then gives

$$\chi_n(t_{m,k}) = (4 - t_{m,k})\chi_{n-1}(t_{m,k}) - \chi_{n-2}(t_{m,k}) \quad \text{for all } n \geq 2.$$

Thus it suffices to show that  $s_{n,m-k} = U_{n-1}\left(\frac{x_{m-k}}{2}\right)$  has this same recursion, i.e.,

$$U_{n-1}\left(\frac{x_{m-k}}{2}\right) = (4 - t_{m,k})U_{n-2}\left(\frac{x_{m-k}}{2}\right) - U_{n-3}\left(\frac{x_{m-k}}{2}\right).$$

From the relation  $4 - t_{m,k} = x_{m-k}$ , the above equation follows immediately from the definition of the Chebyshev polynomials of the first kind via  $U_n(x) := 2xU_{n-1}(x) - U_{n-2}(x)$  for all  $n \geq 2$ .

Thus we have shown (i) is equal to (ii).

The number of recurrent configurations is equal to the determinant of the reduced Laplacian, which by the Matrix-Tree Theorem is equal to the number of spanning trees. We computed this number in 5.0.31.

Now, by performing the *KPW* process on an  $m \times n$  grid, we get a  $(2m+1) \times (2n+1)$  checkerboard with an internal edge removed (and its adjacent edges) as shown in Figure 5.3. Thus we have shown (i) and (iii) are equal. (ii) and (iv) are equal by the sequence in [OEIS].  $\square$

### Remark 5.0.34 (Tilings of odd by odd checkerboards)

Even though there are no ways of tiling an odd by odd checkerboard, it is possible with a missing internal cell. This result is interesting because it gives us a way of counting the number of tilings on an odd by odd checkerboard with a square removed.

We give an example of a tiling of a  $9 \times 9$  checkerboard with a square removed in Figure 5.4:

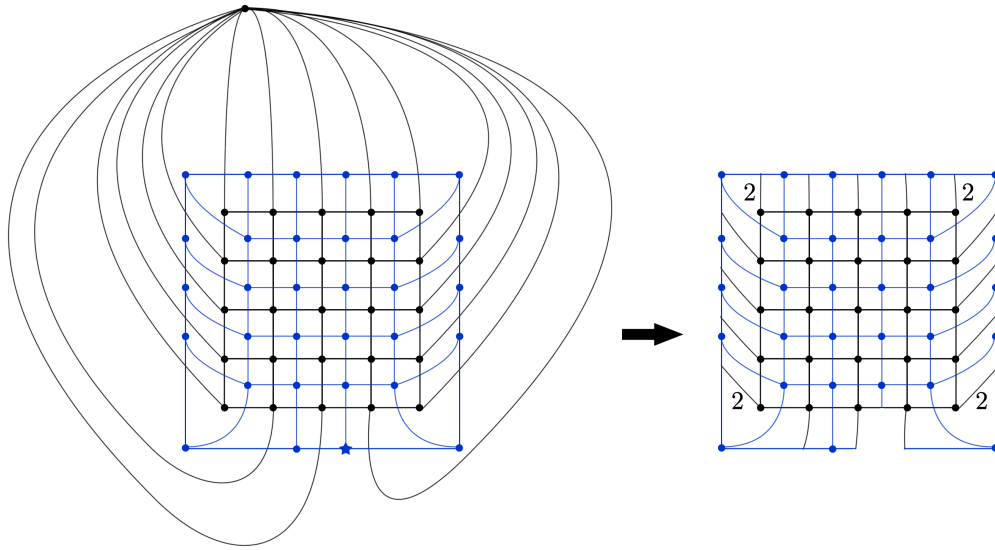


Figure 5.3: Overlaying the dual of the graph we arrive at a graph with a cell removed.

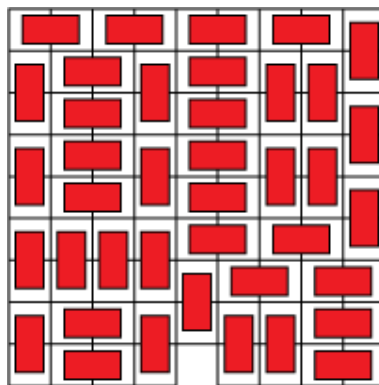


Figure 5.4: Tiling of a  $9 \times 9$  checkerboard with a square removed.



# Conclusion and Open questions

We have investigated the Klein Four group and dihedral symmetry on sandpile grid graphs and related the symmetric recurrents to domino tilings. The main process we applied can be summarized as follows:

- Construct the reduced Laplacian for the sandpile graph. In some cases, this meant the symmetric reduced Laplacian. Furthermore, we know that the determinant of the reduced Laplacian is the number of symmetric recurrents.
- By taking the symmetric reduced Laplacian or transposing it, we get the ordinary reduced Laplacian of another graph, whose number of spanning trees is equal to the determinant of the reduced Laplacian.
- Furthermore, by the process we called KPW, we can associate spanning trees to perfect matchings, and perfect matchings to domino tilings.

Thus, we have seen how Möbius strip tilings and Ciucu graph weighted tilings arise from counting the number of symmetric recurrents on grid graphs.

Our main new theorems relate:

- (i) The number of  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetric recurrents on an even by even grid graph and the number of domino tilings on the even by even checkerboard.
- (ii) The number of  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetric recurrents on an even by odd grid graph and the number of domino tilings on the even by even Möbius checkerboard, as well as the number of weighted domino tilings on an even by even checkerboard.
- (iii) The number of spanning trees on any  $m \times n$  sandpile grid graph and the number of spanning trees on the  $(m + 1) \times (n + 1)$  grid graph.
- (iv) Further, the number of recurrents on any grid graph and the number of domino tilings on an odd by odd checkerboard with an internal edge (and the adjacent edges) removed, with different base cases.
- (v) The number of recurrents on any  $m \times n$  grid graph with modified firings and the number of domino tilings on the  $(2m + 1) \times (2n + 1)$  checkerboard.
- (vi) The number of dihedral symmetric recurrents on an even by even grid graph to the weighted domino tilings on the Ciucu checkerboard.

For most of these, we were able to also give closed formulas involving Chebyshev polynomials.

We have three main questions unanswered.

**Question 6.0.35** *Is there a combinatorial bijection between tilings on the Möbius strip and weighted tilings on the checkerboard as in Theorem 3.0.26?*

In order to have a complete new proof of the number of tilings on the Möbius strip, we need to find a bijection between (i) and (v) in Theorem 3.0.26.

**Question 6.0.36** *The bijections between recurrent elements, spanning trees, and domino tilings induce an addition of domino tilings. Can one easily describe this addition of domino tilings?*

There are several known bijections between recurrent configurations and spanning trees, the most famous of which is the Dhar burning algorithm, as described in [DD90]. However, none of these bijections seem to help us better understand the tiling law.

**Question 6.0.37** *It would be interesting to have a closed formula of the number of weighted domino tilings on the Ciucu graphs. This would involve calculating the determinant of block tridiagonal matrices with blocks of decreasing size. Molinari's technique cannot be applied in this case, because the blocks are not of the same size. Is there a way to calculate that determinant?*

For example, the reduced Laplacian matrix for a  $10 \times 10$  sandpile grid graph with dihedral symmetry looks like:

$$\begin{pmatrix} 4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -3 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 3 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 4 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 \end{pmatrix}$$

You can notice the block matrices on the diagonal having size  $5 \times 5$ ,  $4 \times 4$ ,  $3 \times 3$ ,  $2 \times 2$  and 1.

# Appendix A

## Chebyshev Polynomials

The following is a vignette of Chebyshev polynomials, the tools for finding several closed formulas in this paper.

The *Chebyshev polynomials of the first kind*,  $T_n$ , are defined by the recurrence relation:

$$\begin{aligned}T_0(x) &= 1 \\T_1(x) &= x \\T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x).\end{aligned}$$

They arise in the computation of the determinant of the following matrix:

$$T_n(x) = \det \begin{pmatrix} x & 1 & & & \\ 1 & 2x & 1 & & \\ & 1 & 2x & & \\ & & \ddots & \ddots & \\ & & & 1 & 2x \end{pmatrix}.$$

The *Chebyshev polynomials of the second kind*,  $U_n$ , are defined by the recurrence relation:

$$\begin{aligned}U_0(x) &= 1 \\U_1(x) &= 2x \\U_{n+1}(x) &= 2xU_n(x) - U_{n-1}(x).\end{aligned}$$

They arise in the computation of the determinant of the following matrix:

$$U_n(x) = \det \begin{pmatrix} 2x & 1 & & & \\ 1 & 2x & 1 & & \\ & 1 & 2x & & \\ & & \ddots & \ddots & \\ & & & 1 & 2x \end{pmatrix}.$$

The two kinds of Chebyshev polynomials are related by the following equations:

$$\begin{aligned}T_n(x) &= \frac{1}{2}(U_n(x) - U_{n-2}(x)) \\T_n(x) &= U_n(x) - xU_{n-1}(x)\end{aligned}$$

The roots of  $T_n$  are  $x_k = \cos \frac{(2k-1)\pi}{2n}$ ,  $k = 1, \dots, n$ .

Similarly, the roots of  $U_n$  are  $x_k = \cos \frac{k\pi}{n+1}$ ,  $k = 1, \dots, n$ .

# Appendix B

## Determinants of Block Tridiagonal Matrices

The theorems in chapter 3 make use of a special way of computing the determinant of block tridiagonal matrices. We introduce a theorem from [Mol08] which will be helpful for our proofs.

First, let  $M^{(0)}$  be

$$\begin{pmatrix} A_1 & B_1 & & 0 \\ C_1 & \ddots & \ddots & \\ & \ddots & \ddots & B_{n-1} \\ 0 & & C_{n-1} & A_n \end{pmatrix}.$$

**Theorem B.0.38**  $\det M^{(0)} = (-1)^{nm} \det [T_{11}^{(0)}] \det [B_1 \cdots B_n - 1]$ ,

where  $T_{11}^{(0)}$  is the upper left block of size  $m \times m$  of the transfer matrix

$$T^{(0)} = \begin{bmatrix} -A_n & -C_{n-1} \\ I_m & 0 \end{bmatrix} \begin{bmatrix} -B_{n-1}^{-1}A_{n-1} & -B_{n-1}^{-1}C_{n-2} \\ I_m & 0 \end{bmatrix} \cdots \begin{bmatrix} -B_1^{-1}A_1 & -B_1^{-1} \\ I_m & 0 \end{bmatrix}. \quad (\text{B.1})$$

**Proof.** The linear system  $M^{(0)}\Psi = 0$  can be solved through the transfer matrix technique, by taking:

$$\begin{bmatrix} \psi_n \\ -C_{n-1}^{-1}A_n\psi_n \end{bmatrix} = \begin{bmatrix} -B_{n-1}^{-1}A_{n-1} & -B_{n-1}^{-1}C_{n-2} \\ I_m & 0 \end{bmatrix} \cdots \begin{bmatrix} -B_2^{-1}A_2 & -B_2^{-1}C_1 \\ I_m & 0 \end{bmatrix} \begin{bmatrix} -B_1^{-1}A_1\psi_1 \\ \psi_1 \end{bmatrix}. \quad (\text{B.2})$$

Now if we multiply on the right by the nonsingular matrix

$$\begin{bmatrix} -A_n & -C_{n-1} \\ I_m & 0 \end{bmatrix}$$

and we rewrite the right-hand vector as the product

$$\begin{bmatrix} -B_1^{-1}A_1 & -B_1^{-1} \\ I_m & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ 0 \end{bmatrix},$$

we can transform B.2 into an equation for the transfer matrix  $T^{(0)}$ , connecting the boundary components with  $\psi_{n+1} = 0$  and  $\psi_0 = 0$  :

$$\begin{bmatrix} 0 \\ \psi_n \end{bmatrix} = T^{(0)} \begin{bmatrix} \psi_1 \\ 0 \end{bmatrix}. \quad (\text{B.3})$$

Equation B.3 implies that  $\det T_{11}^{(0)} = 0$ , which is dual to  $\det M^{(0)} = 0$ . This can be transformed into an identity by introducing a parameter  $\lambda$  and comparing the polynomials  $\det[\lambda I_{nm} - M^{(0)}]$  and  $\det T^{(0)}(\lambda)$ , which are obtained by replacing the blocks  $A_i$  with  $A_i - \lambda I_m$ . The polynomials are proportional because both are polynomials in  $\lambda$  of degree  $nm$  and with the same roots. The constant is fixed by their behavior for large  $\lambda$ .  $\square$

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