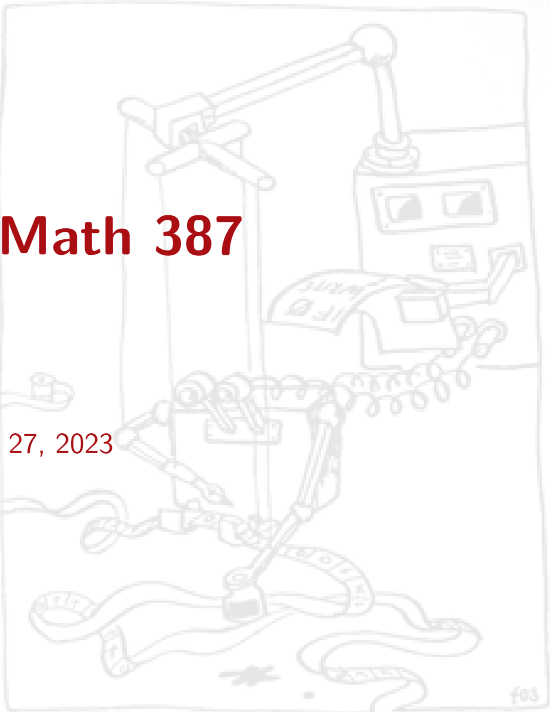

CS/Math 387

February 27, 2023

Deciding whether a Turing machine language is regular

Let $\text{REGULAR}_{TM} = \{\langle M \rangle : M \text{ is a TM and } L(M) \text{ is regular}\}$.

Theorem 5.3. REGULAR_{TM} is undecidable.

Proof. Suppose R decides REGULAR_{TM}. We will use it to construct a TM S that decides the undecidable language A_{TM}:

$S =$ "On input $\langle M, w \rangle$, where M is a TM and w is a string:

1. Construct the following TM N.
2. $N =$ "On input x:
   1. If x has the form $0^n 1^n$ for some $n \geq 0$, then **accept**.
   2. If x does not have this form, run M on input w and accept if M accepts w."
3. Run R on input $\langle N \rangle$.
4. If R accepts, then **accept**; if R rejects, then **reject**."

R accepts $\Leftrightarrow L(N) = \Sigma^* \Leftrightarrow w$ accepted by M;
R rejects $\Leftrightarrow L(N) = \{0^n 1^n : n \geq 0\} \Leftrightarrow w$ not accepted by M.

Deciding whether Turing machine's language reversible

Let $T = \{\langle M \rangle : M$ is a TM and $L(M) = L(M)^R\}$.
Then T is undecidable.

Proof. Suppose R decides T. We will use it to construct a TM S that decides the undecidable language A_{TM}:

$S =$ "On input $\langle M, w \rangle$, where M is a TM and w is a string:

1. Construct the following TM N.
2. $N =$ "On input x:
    1. If $x = 01$, then **accept**.
    2. If $x \neq 01$ run M on input w and accept if M accepts w."
3. Run R on input $\langle N \rangle$.
4. If R accepts, **accept**; if R rejects, **reject**."

R accepts $\Leftrightarrow L(N) = \Sigma^* \Leftrightarrow w$ accepted by M;
R rejects $\Leftrightarrow L(N) = \{01\} \Leftrightarrow w$ not accepted by M.

Rice's theorem

A *property* P of recognizable languages is a set of Turing-recognizable languages.

A property P is *non-trivial* if $P \neq \emptyset$ and P is not the set of all recognizable languages.

Rice's theorem. Let P be a non-trivial property of recognizable languages. Then the language

$$P_{TM} = \{\langle M \rangle : M \text{ is a Turing machine with } L(M) \in P\}$$

is undecidable.

Proof of Rice's theorem

To prove: $P_{TM} = \{\langle M \rangle : M \text{ is a Turing machine with } L(M) \in P\}$ is undecidable.

Proof. Suppose P_{TM} is decidable by a TM R.

To get a contradiction, we will use R to construct a TM deciding A_{TM}.

We may assume $\emptyset \notin P$.

Reason: If $\emptyset \in P$, then consider

$$\overline{P} = \{A : A \text{ is recognizable and } A \notin P\}.$$

Then, $\emptyset \notin \overline{P}$; and $\overline{P}$ is non-trivial; Let

$$\overline{P}_{TM} = \{\langle M \rangle : M \text{ is a Turing machine with } L(M) \in \overline{P}\}.$$

$\overline{P}_{TM}$ is decidable if and only if P_{TM} is decidable. We would then let R be a decider for $\overline{P}_{TM}$.

Proof of Rice's theorem

To prove: $P_{TM} = \{\langle M \rangle : M \text{ is a Turing machine with } L(M) \in P\}$ is undecidable.

Proof. Suppose P_{TM} is decidable by a TM R.

To get a contradiction, we will use R to construct a TM deciding A_{TM}.

We may assume $\emptyset \notin P$.

Since P is non-trivial, there exists some language $A \in P$.

Say A is recognized by a TM M_A.

Proof of Rice's theorem

- P_{TM} is decidable by a TM R.
- $\emptyset \notin P$, $A \in P$, recognized by TM M_A.

Given $\langle M, w \rangle$, construct a TM N:

$N =$ "On input x:

1. Simulate M on w.
2. If M accepts w, then simulate M_A on x.
3. If M_A accepted x, then **accept**.

"

If M accepts w, then $L(N) = A$.
If M does not accept w, then $L(N) = \emptyset$.

Proof of Rice's theorem

$N =$ "On input x:

1. Simulate M on w.
2. If M accepts w, then simulate M_A on x.
3. If M_A accepted x, then **accept**.

"

If M accepts w, then $L(N) = A$; if not, $L(N) = \emptyset$.

Decider H for A_{TM}:

$H =$ "On input $\langle M, w \rangle$:

1. Construct N from $\langle M, w \rangle$.
2. Feed $\langle N \rangle$ to R, the decider for P_{TM}.
3. If R accepts, then **accept**; if R rejects, then **reject**.

"

R decides $P_{TM} = \{\langle M \rangle : M \text{ is a TM and } L(M) \in P\}$.

Applicability of Rice's theorem

$T = \{\langle M \rangle : M \text{ is a TM and } L(M) = L(M)^R\}$ is undecidable.

Proof. Define $P = \{A : A \text{ recognizable and } A = A^R\}$. The P is a property of recognizable languages, and P is non-trivial, e.g., $\{01, 10\} \in P$ and $\{01\} \notin P$. The result follows from Rice's theorem.

Applicability of Rice's theorem

$$H = \{\langle M \rangle : M \text{ is a TM and } M \text{ never writes a blank over a non-blank symbol}\}.$$

Rice's theorem is not applicable since

$$\begin{aligned} P &= \{A : A \text{ recognizable by a TM } M \text{ that never writes a blank over a non-blank}\} \\ &= \{\text{all recognizable languages}\} \end{aligned}$$

Applicability of Rice's theorem

$EQ_{\mathrm{TM}} = \{\langle M, N\rangle : M \text{ and } N \text{ are TMs and } L(M) = L(N)\}$.

Rice's theorem is not applicable because the property here concerns a pair of recognizable languages.

See the link on our course homepage.