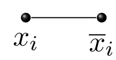
# CS/Math 387 Homework for Friday, Week 9

Problem 1. An undirected finite graph is 3-*colorable* if there is a way to assign one of three different colors to each of its vertices in such a way the no two adjacent vertices have the same color. Let

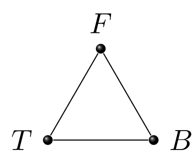
$$3\text{COLOR} = \{\langle G \rangle : G \text{ is a 3-colorable graph}\}.$$

In this problem, you are asked to give a polynomial time reduction of 3SAT to 3COLOR in order to show that 3COLOR is NP-complete.

To describe the reduction, let the colors be True, False, and Blank. Take one vertex for each literal $x_i$ and one for its negation $\overline{x}_i$. We do not want both $x_i$ and $\overline{x}_i$ to both be assigned True or both be assigned False. So add edges:

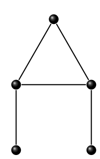



To make sure that no vertex corresponding to a literal is colored Blank, add the following triangle to the graph:




The vertices are labeled $T$, $F$, and $B$, as shown. (These are labels, not the colors True, False, Blank. However, given any 3-coloring, we can permute the colors to get another 3-coloring such that $T, F, B$ are colored True, False, Blank, respectively). Now connect vertices representing literals to the vertex of the triangle labeled by $B$.

The following structure serves as an "OR-gadget":



By adding two edges from the top vertex of the gadget: one to vertex $B$ and one to vertex $F$, we can assure that in any 3-coloring, the top vertex will have the same color as vertex $T$.

(a) Use the pieces above to create a graph corresponding to the clause $x \vee \overline{y}$ in the variables $x$ and $y$.

(b) Use the pieces above to create a graph corresponding to the clause $x \vee \overline{y} \vee z$ in the variables $x, y$ and $z$.

(c) Describe the construction for a general formula in 3SAT.

Problem 2. A $k$-*coloring* of a graph $G$ is an assignment of one of $k$ colors to each of the vertices of $G$ such that no edge has endpoints of the same color. We have seen that the 3-coloring problem is NP-complete. Here we prove that the $k$-coloring problem is also NP-complete for $k \geq 3$ by reduction from 3-coloring. Let $k \geq 3$. Given $G$, form a new graph $H$ by first taking the disjoint union of $G$ with $K_{k-3}$, the complete graph on $k-3$ vertices. Then add edges to $H$ connecting every vertex of $K_{k-3}$ to every vertex of $G$. Prove that $G$ is 3-colorable if and only if $H$ is $k$-colorable. (Prove both implications even though only one is necessary for the reduction.)

PROBLEM 3. Let $f_1, \ldots, f_m$ be a list of final exams, and let $s_1, \ldots, s_n$ be a list of students. Each student takes some specified subset of the final exams. Each final exam is a single time slot in length. The problem is to determine if these finals can be scheduled in only $h$ non-overlapping time slots so that no student has two simultaneous exams. Show that this scheduling problem is NP-complete, in general, by reducing from $k$-coloring of graphs for some $k$.

PROBLEM 4. Sipser described a polynomial time reduction of 3SAT to CLIQUE.

(a) (i) Use the same idea to associate a graph to the boolean formula $\phi = (x \vee y) \wedge (\overline{x} \vee z) \wedge (y \vee \overline{z})$ so that the solutions in the formula correspond to 3-cliques in the graph.
  (ii) Give some specific examples of solutions and their corresponding cliques.

(b) Suppose you are given a 3SAT problem with $k$ clauses. Construct the corresponding graph as described by Sipser. Is it possible for the graph to have a clique of size larger than $k$. Give a proof that is it not possible, or give an explicit example showing that it is possible.

PROBLEM 5. Show VCOVER is NP-complete by giving a polynomial time reduction from CLIQUE. So given a graph $G$ and a number $k$, you need to construct an instance of VCOVER, a graph $G'$ and a number $m$, so that finding $G$ has a $k$-clique if and only if $G'$ has an $m$-vertex cover.

Hint: Let $G'$ *complement* of $G$. This means $G'$ and $G$ have the same vertex set $V$ but $\{u, v\}$ is an edge of $G'$ if and only if $\{u, v\}$ is not an edge of $G$. Once you have a well-formulated reduction, separate your if-and-only-if proof into two parts.

PROBLEM 6. Prove that PSPACE is closed under each of the following operations:

(a) Union.

(b) Concatenation.