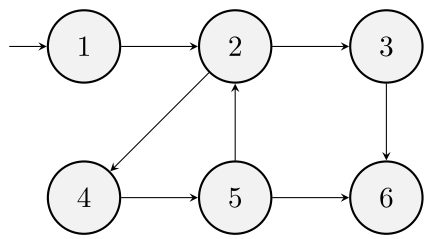
PROBLEM 1.

(a) Given a language B in PSPACE, word $u \in B$, a word $v \notin B$, and a language $A \in$ PSPACE, describe a TM M running in polynomial space that computes a function $f\colon \Sigma^* \to \Sigma^*$ such that $w \in A$ if and only if $f(w) \in B$. Thus, on input w, M halts with $f(w)$ on its tape. We write $A \leq_{\text{PSPACE}} B$. This problem shows that everything in PSPACE is reducible to any other "nontrivial" element of PSPACE.

(b) Does the same argument work if we replace PSPACE with P? You are given a nontrivial $B \in$ P and another language in $A \in$ P. Can you compute a mapping f in polynomial time and conclude that $A \leq_m B$.

(c) Why are we interested polynomial time reductions and not polynomial space reductions? (Relevant facts: P $\subseteq$ PSPACE, and PSPACE = NSPACE.)

PROBLEM 2. Consider the following generalized geography game where the start node is the one with the arrow pointing in from nowhere. Does Player I have a winning strategy? Does Player II? Give reasons for your answers. (To start, Player I is at node 1 and gets to choose the next node.)



PROBLEM 3.

(a) Encode the formula

$$\exists x_1 \forall x_2 \exists x_3 [(x_1 \vee x_2) \wedge (x_2 \vee x_3) \wedge (\overline{x_2} \vee \overline{x_3})]$$

as a game of generalized geography using the method described in our book. The players are Player $\exists$ and Player $\forall$. Player $\exists$ starts at the first node and gets to decide the next node.

(b) State a winning strategy for one of the players in the geography game, and describe how this strategy gives the truth value of the formula.

PROBLEM 4.

(a) Perform the following multiplication problem of two binary numbers using the standard algorithm from grade school:

$$\begin{array}{r} 101101 \\ \text{x} \quad 1011 \\ \hline \end{array}$$

This exercise will help you think about the next part of this problem.

(b) Let

$$\text{MULT} = \{a\#b\#c : a, b, c \text{ are binary natural numbers and } a \times b = c\}.$$

Show that MULT $\in$ L. [Hints: Storing the whole table of values created by standard multiplication cannot be done in log space. However, you are allowed to use a fixed number of pointers to the read-only tape and an additional $O(\log(n))$ space on the work tape (why?). Storing the carry might be a worry, but what is the maximum size of the carry compared to the length of a and b?]

PROBLEM 5. (bonus) The cat-and-mouse game is played by two players, "Cat" and "Mouse", on an arbitrary undirected graph. At a given point, each player occupies a node of the graph. The players take turns moving to a node adjacent to the one that they currently occupy. A special node of the graph is called "Hole". Cat wins if the two players ever occupy the same node. Mouse wins if it reaches the Hole before the preceding happens. The game is a draw if a situation repeats (i.e., the two players simultaneously occupy positions that they simultaneously occupied previously, and it is the same player's turn to move). Define

$$\begin{aligned}\text{HAPPY-CAT} = \{\langle G, c, m, h\rangle \mid\ & G, c, m, h \text{ are respectively a graph, and positions} \\ & \text{of the Cat, Mouse, and Hole, such that Cat} \\ & \text{has a winning strategy if Cat moves first}\}.\end{aligned}$$

Show that HAPPY-CAT is in P. (Hint: The solution is not complicated and doesn't depend on subtle details in the way the game is defined. Consider the entire game tree. It is exponentially big, but you can search it in polynomial time. To get a feel for the solution, start by listing several obvious winning positions.)