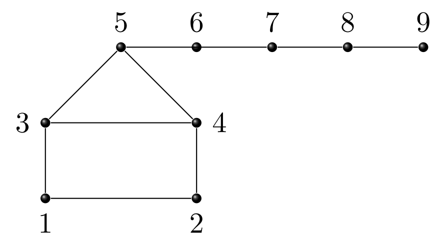
PROBLEM 1. Let G be a simple undirected graph. A *vertex cover* of G is a set of vertices of G such that each edge of G contains at least one vertex from the set. A *minimal vertex cover* is a vertex cover of minimum cardinality. Define the language

$$\text{VCOVER} = \{\langle G, m\rangle : G \text{ is a graph which has a vertex cover of size } m\}.$$

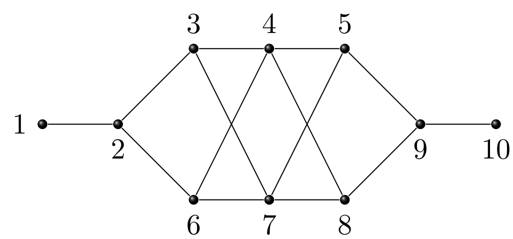
(a) Let $n \geq 2$. Find a minimal vertex cover for the complete graph K_n with vertices $(1, \ldots, n)$. Prove that your set covers and that any smaller set would not.

(b) Let $n \geq 3$. Find a minimal vertex cover for the cycle graph C_n with vertices $\{1, \ldots, n\}$ arranged in a cycle. (No explanation necessary.)

(c) Find a minimal vertex cover for following graph (no explanation required):



(d) Find a minimal vertex cover for following graph (no explanation required):

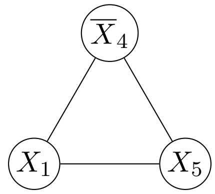


(e) Show VCOVER is NP-complete by giving a polynomial time reduction to CLIQUE. So given a graph G and a number k, you need to construct an instance of VCOVER—a graph G' and a number m—so that finding G has a k-clique if and only if G' has an m-vertex cover.

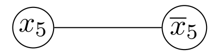
Hint: Let G' be the *complement* of G. This means G' and G have the same vertex set V but $\{u, v\}$ is an edge of G' if and only if $\{u, v\}$ is not an edge of G. Once you have a well-formulated reduction, separate your if and only if proof into two parts.

Start with some examples!

PROBLEM 2. Here is an attempt at a polynomial time reduction of 3SAT to VCOVER. The gadgets are hinted at below. The goal is to create graph that has a minimal vertex cover of a certain size if and only if its corresponding 3SAT formula is satisfiable.



clause gadget



variable gadget

(a) Encode the clause $x_1 \vee \overline{x}_4 \vee x_5$ as a graph using 9 vertices.

(b) What is the size of a minimal vertex cover of your graph?

(c) Encode the clause $(x_1 \vee \overline{x}_4 \vee x_5) \wedge (x_2 \vee x_4 \vee x_5)$ as a graph using 14 vertices.

(d) What is the size of a minimal vertex cover of your graph?

(e) A general 3SAT formula ϕ may be encoded as an instance of the minimal vertex color problem by generalizing the above construction. You do not need to describe it here. However, answer this: if ϕ has k clauses, and the total number of variables appearing in ϕ is ℓ, what is the size of a minimal vertex color in the case the formula is satisfiable? (A variable has the form x_i, and we count it as appearing in ϕ if it or its negation, $\overline{x}_i$ is in any clause of ϕ. Thus, for instance, the total number of variables appearing in $(x_1 \vee \overline{x}_4 \vee x_5) \wedge (x_2 \vee x_4 \vee x_5)$ is 4.)

(f) Describe how a polynomial time decider for VCOVER would provide a polynomials time decider for 3SAT