Problem 1.

(a) Find a match in the following instance of the Post Correspondence Problem (PCP):

$$\left\{ \left[\frac{a}{baa}\right], \left[\frac{ab}{aa}\right], \left[\frac{bba}{bb}\right] \right\}.$$

(b) We have seen that PCP is undecidable. Prove that $\overline{\text{PCP}}$ is unrecognizable.

Problem 2. We start with some definitions. A *property* P of recognizable languages is a set of Turing-recognizable languages. In other words, for each $A \in P$ there must exist some TM M such that $L(M) = A$. We say P is *non-trivial* if (i) it is non-empty, and (ii) there exists some recognizable language B such that $B \notin P$. We say a TM M *has property* P, if $L(M) \in P$.

Rice's theorem. Let P be a non-trivial property of recognizable languages. Then the language

$$P_{TM} = \{\langle M\rangle : M \text{ is a Turing machine with } L(M) \in P\}$$

is undecidable.

(a) Give an example of a set of languages that does not form a property of Turing recognizable languages.

(b) Indicate, with explanation, which of the following are properties of recognizable languages:

$$\text{(i) } \{\emptyset\}, \qquad \text{(ii) } \{\Sigma^*\}, \qquad \text{(iii) } \{\text{all decidable languages}\}.$$

(c) Give all examples of trivial properties of recognizable languages.

Problem 3. Decide whether Rice's theorem applies to the following languages. If so, formally define the property (i.e., $P = \{\text{blah}\}$) and show it is non-trivial by giving two examples of recognizable languages—one in P and one not in P. Conclude the language is not decidable. If Rice's theorem does not apply, explain why the property in question is not a property of recognizable languages or is trivial.

(a) $T = \{\langle M\rangle : M \text{ is a TM and } L(M) = L(M)^R\}$.

(b) $R = \{\langle M\rangle : M \text{ is a TM and } L(M) \text{ is regular}\}$.

(c) $S = \{\langle M\rangle : M \text{ is a TM and } L(M) = \{\langle M\rangle\}\}$. (Be careful/precise with this one.)

(d) $H = \{\langle M\rangle : M \text{ is a TM and } M \text{ never writes a blank over a non-blank symbol}\}$.

Problem 4. Let $T = \{\langle M\rangle : M \text{ is a TM and } L(M) = L(M)^R\}$. Describe a reduction from A_{TM} in order to prove that T is undecidable. (Hint: mimic the proof of Theorem 5.3 in our text.)