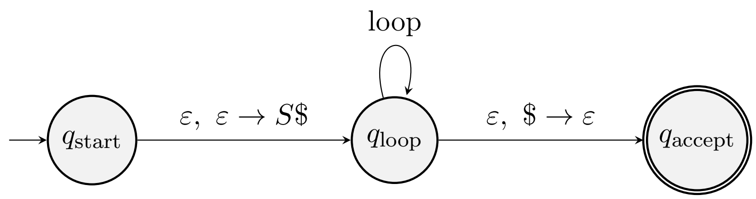
PROBLEM 1. Let A be the following CFG from our text (Example 2.25):

$$S \to aTb|b$$
$$T \to Ta|\varepsilon.$$

The text describes the construction for the following PDA that recognizes $L(A)$:



The loop has the following transitions:

$$\begin{aligned} &\varepsilon,\ S \to aTb|b \\ &\varepsilon,\ T \to Ta|\varepsilon \\ &a,\ a \to \varepsilon \\ &b,\ b \to \varepsilon. \end{aligned}$$

Here, for instance, the notation $\varepsilon,\ S \to aTb|b$ is shorthand for two transitions: $\varepsilon,\ S \to aTb$ and $\varepsilon,\ S \to b$. Also, for example, the transition $\varepsilon,\ S \to aTb$, means that if S is at the top of the stack, one can pop S and push aTb without reading any input.

(a) Describe $L(A)$.

(b) Show how the PDA accepts $aaab$ one step at a time. List the new state, the transition, and the stack. The first step is

transition	$q_{\text{start}} \xrightarrow{\varepsilon,\varepsilon\to S\$} q_{\text{loop}}$
stack	empty $\longrightarrow$ $S\$$

Solution.

(a) $L(A)$ is described by the regular expression a^*b.

(b)

$$\begin{array}{ccccccccc} q_{\text{start}} & \xrightarrow{\varepsilon,\varepsilon\to S\$} & q_{\text{loop}} & \xrightarrow{\varepsilon,\ S\to aTb} & q_{\text{loop}} & \xrightarrow{\varepsilon,\ a\to\varepsilon} & q_{\text{loop}} & \xrightarrow{\varepsilon,\ T\to Ta} & q_{\text{loop}} \\ \hline \text{empty} & \longrightarrow & S\$ & \longrightarrow & aTb\$ & \longrightarrow & Tb & \longrightarrow & Tab\$ \end{array}$$

$$\begin{array}{ccccccccc} q_{\text{loop}} & \xrightarrow{\varepsilon,\ T\to Ta} & q_{\text{loop}} & \xrightarrow{\varepsilon,\ T\to\varepsilon} & q_{\text{loop}} & \xrightarrow{a,\ a\to\varepsilon} & q_{\text{loop}} & \xrightarrow{a,\ a\to\varepsilon} & q_{\text{loop}} \\ \hline Tab\$ & \longrightarrow & Taab\$ & \longrightarrow & aab\$ & \longrightarrow & ab\$ & \longrightarrow & b\$ \end{array}$$

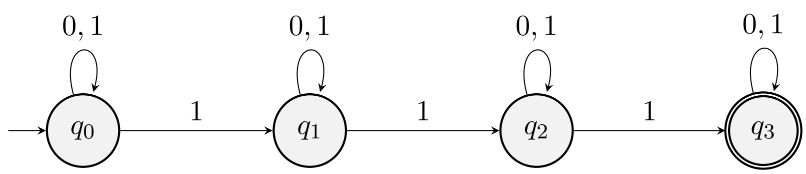
$$\begin{array}{ccccc} q_{\text{loop}} & \xrightarrow{b,\ b\to\varepsilon} & q_{\text{loop}} & \xrightarrow{\$,\ \$\to\varepsilon} & q_{\text{accept}} \\ \hline b\$ & \longrightarrow & \$ & \longrightarrow & \text{empty.} \end{array}$$

Problem 2. Let $\Sigma = \{0, 1\}$, and consider the language $L = \{w : w \text{ contains at least three 1s}\}$.

(a) Find a regular expression for L.
(b) Convert that regular expression into an NFA recognizing L.
(c) Find a CFG for the language.
(d) Convert your CFG into a PDA using the algorithm in the text (as in Problem 1).

Solution.

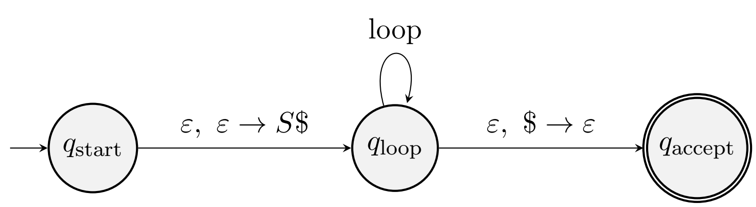
(a) A regular expression for L is $\Sigma^*1\Sigma^*1\Sigma^*1\Sigma^*$.
(b) A corresponding NFA:



(c) The algorithm from gives the corresponding CFG:

$$
\begin{aligned}
S &\to q_0 \\
q_0 &\to 0q_0|1q_0|1q_1 \\
q_1 &\to 0q_1|1q_1|1q_2 \\
q_2 &\to 0q_2|1q_2|1q_3 \\
q_2 &\to 0q_2|1q_2|1q_3 \\
q_3 &\to 0q_3|1q_3|\varepsilon.
\end{aligned}
$$

(d) The corresponding PDA:



The loop has the following transitions:

$$
\begin{aligned}
&\varepsilon,\ S \to Q_0 \\
&\varepsilon,\ Q_0 \to 0Q_0|1Q_0|1Q_1 \\
&\varepsilon,\ Q_1 \to 0Q_1|1Q_1|1Q_2 \\
&\varepsilon,\ Q_2 \to 0Q_2|1Q_2|1Q_3 \\
&\varepsilon,\ Q_2 \to 0Q_2|1Q_2|1Q_3 \\
&\varepsilon,\ Q_3 \to 0Q_3|1Q_3|\varepsilon \\
&0,\ 0 \to \varepsilon \\
&1,\ 1 \to \varepsilon.
\end{aligned}
$$

PROBLEM 3. Let $\Sigma = \{0, 1\}$, and consider the language $L = \{0^m 1^n : m \neq n\}$.

(a) Find a CFG generating L using at most three variables.

(b) Convert your CFG into a PDA using the algorithm in the text (as in Problem 1).

(c) Is L regular? Prove or disprove.

PROBLEM 4. (Bonus) Our goal in this problem is to show that the representation of objects can affect whether or not a given set can be recognized by a machine. Consider a set A of natural numbers. Let $B_k(A)$ be the set of strings that represent numbers from A in base k (with no leading zeros). For example, if $A = \{3, 5\}$ then $B_2(A) = \{11, 101\}$ and $B_3(A) = \{10, 12\}$. We can think of $B_k(A)$ as a language with a k-symbol alphabet. Give a set A for which $B_2(A)$ is regular but $B_3(A)$ is not (and prove it).