PROBLEM 1. For each of the following, prove either that the language is regular or that it is not regular. In all cases $\Sigma = \{0.1\}$.

(a) $L_1 = \{w : w \text{ contains an equal number of 0s and 1s}\}$.

(b) $L_2 = \{1^k w : w \in \Sigma^*, k \geq 1, \text{ the number of 1s in } w \text{ is at least } k\}$.

(c) $L_3 = \{1^k w : w \in \Sigma^*, k \geq 1, \text{ the number of 1s in } w \text{ is at most } k\}$.

PROBLEM 2. Consider the context-free grammar G

$$S \to RR$$
$$R \to 0R1 \mid \varepsilon.$$

(a) Which of the following are in $L(G)$? For each that is, give a left-most derivation making one substitution at each step.
   (i) 001101
   (ii) 000111.
   (iii) 1010.
   (iv) ε.

(b) Succinctly describe the language $L(G)$. Explain your reasoning.

(c) Is $L(G)$ regular? Provide a proof.

Solution.

(a) (i) $001101 \in L(G)$:

$$S \to RR \to 0R1R \to 00R11R \to 0011R \to 00110R1 \to 001101.$$

   (ii) $000111 \in L(G)$:

$$S \to RR \to R \to 0R1 \to 00R11 \to 000R111.$$

   (iii) $1010 \notin L(G)$.

   (iv) $\varepsilon \in L(G)$:

$$S \to RR \to R \to \varepsilon.$$

(b) The language produced by the CFG

$$S \to R$$
$$R \to 0R1|\varepsilon$$

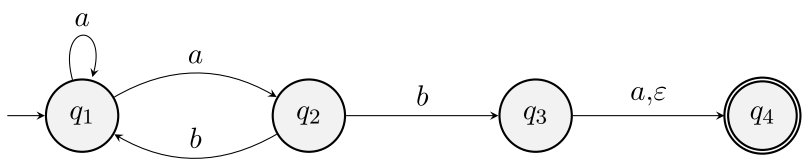
is $A := \{0^k 1^k : k \geq 0\}$. The production $S \to RR$ in G then says that $L(G) = A^2 = AA := \{ww : w \in A\}$.

(c) The language $L(G)$ is not regular. We prove this by contradiction. If it we regular, then take a pumping length p and consider the word $0^p1^p \in L(G)$. By the pumping lemma $0^p1^p = xyz$ where $|xy| \leq p$, $|y| > 0$, and $xy^iz \in L(G)$ for all $i \geq 0$. Then y consists of all 0s, and pumping it would produce words in $L(G)$ that have more 0s than 1s, which is not possible.

PROBLEM 3. Let A be the language of all words in $\Sigma = \{0, 1\}$ that contain the same number of occurrences of 01 as occurrences of 10. Is A regular? Provide a proof.

Solution. The language A is regular. In fact, A is given by the regular expression $(0^+1^+0^+)^* \cup (1^+0^+1^+)^*$.

PROBLEM 4. Let A be the NFA pictured below:



(a) Create a CFG G such that $L(A) = L(G)$.

(b) Give a derivation of the word $aabab$ making one substitution at each step.

PROBLEM 5. Let $P = \{0^q : q \text{ is a prime number}\}$. Prove or disprove that P is regular.

PROBLEM 6. (Bonus) Our goal in this problem is to show that the representation of objects can affect whether or not a given set can be recognized by a machine. Consider a set A of natural numbers. Let $B_k(A)$ be the set of strings that represent numbers from A in base k (with no leading zeros). For example, if $A = \{3, 5\}$ then $B_2(A) = \{11, 101\}$ and $B_3(A) = \{10, 12\}$. We can think of $B_k(A)$ as a language with a k-symbol alphabet. Give a set A for which $B_2(A)$ is regular but $B_3(A)$ is not (and prove it).