Tree inversions and parking functions

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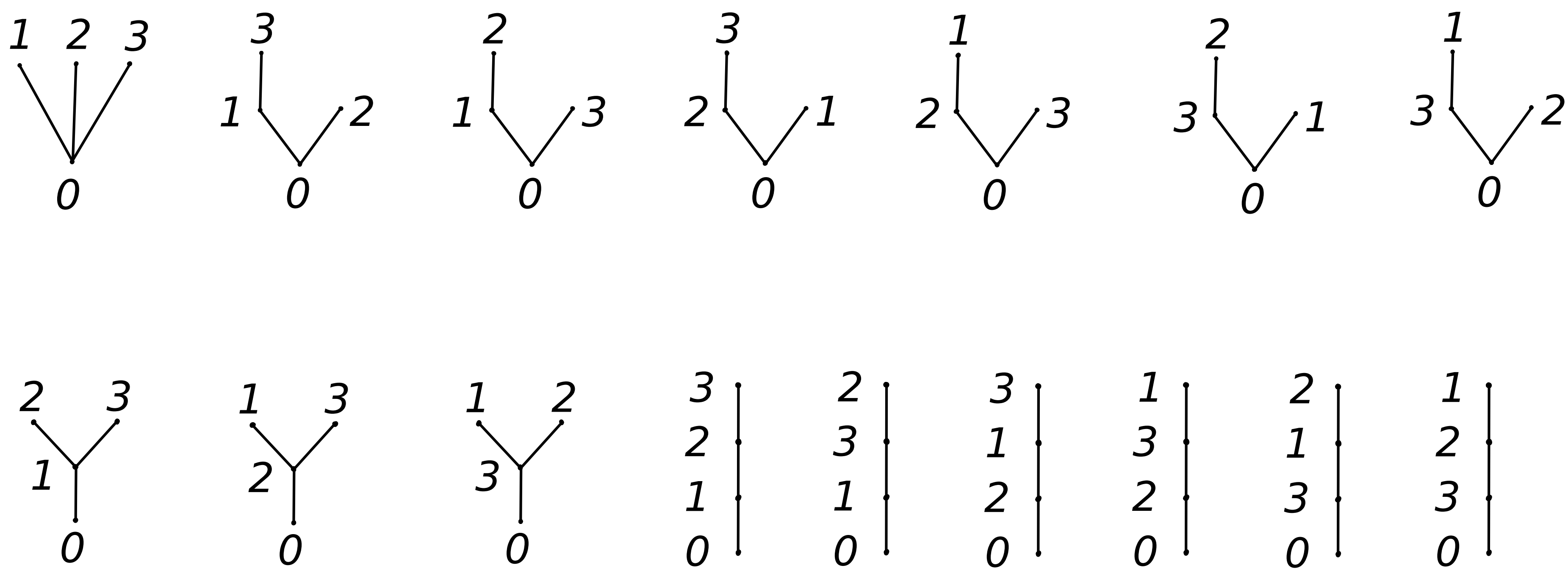
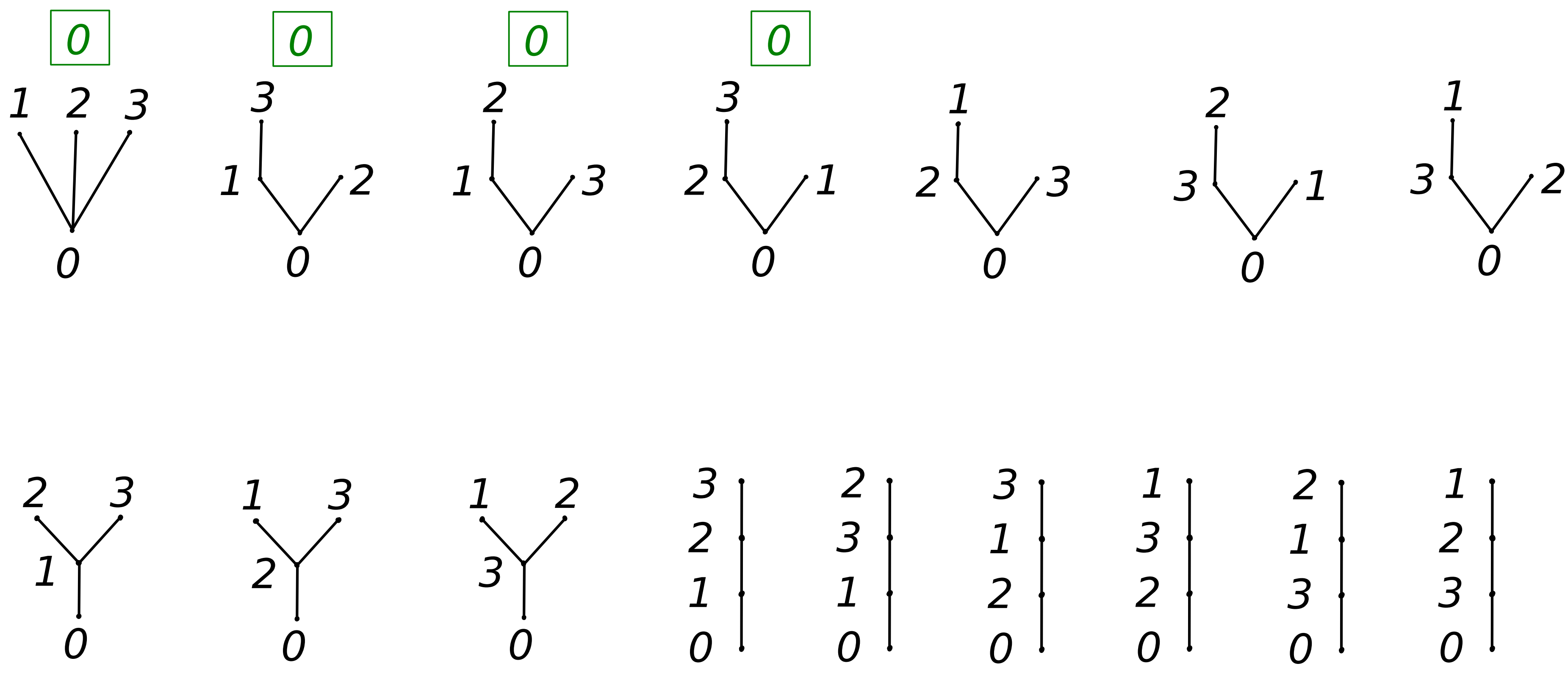
Rooted labeled trees on $n+1$ vertices

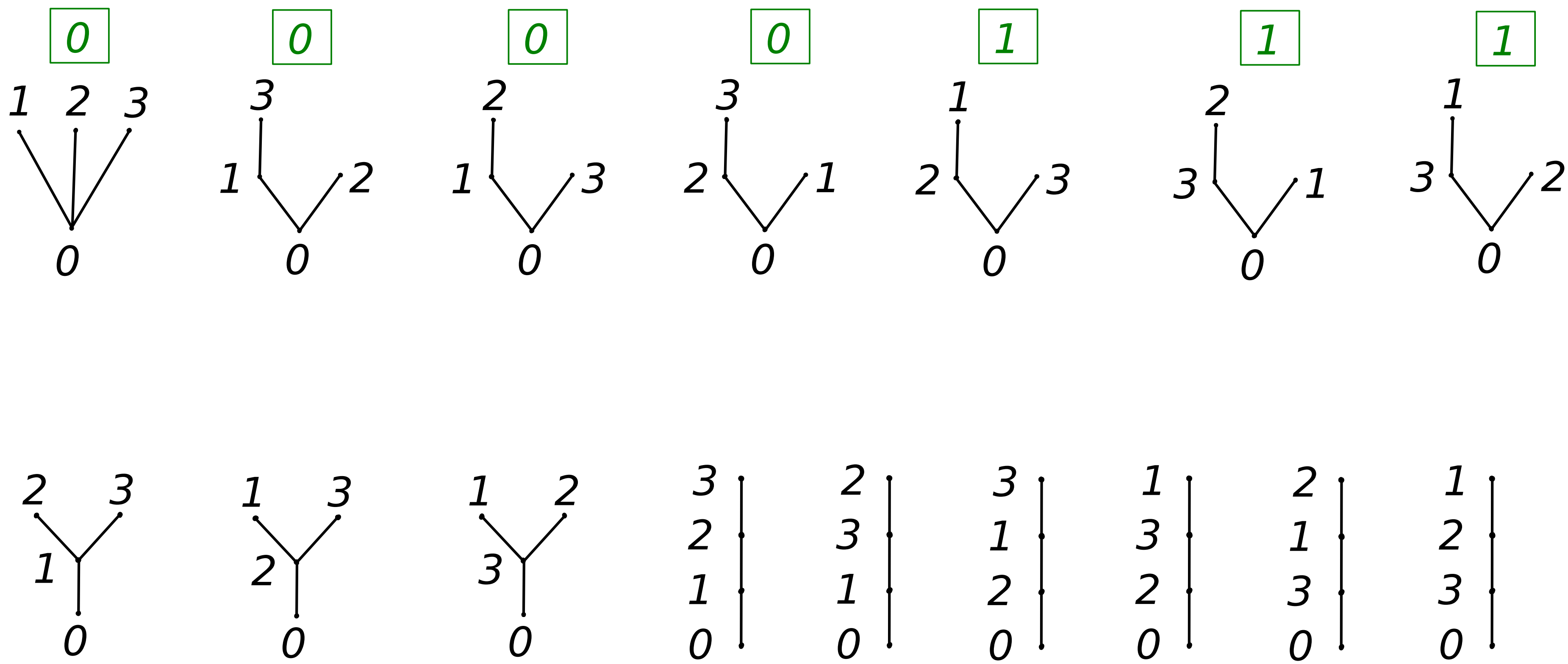
$n = 3$



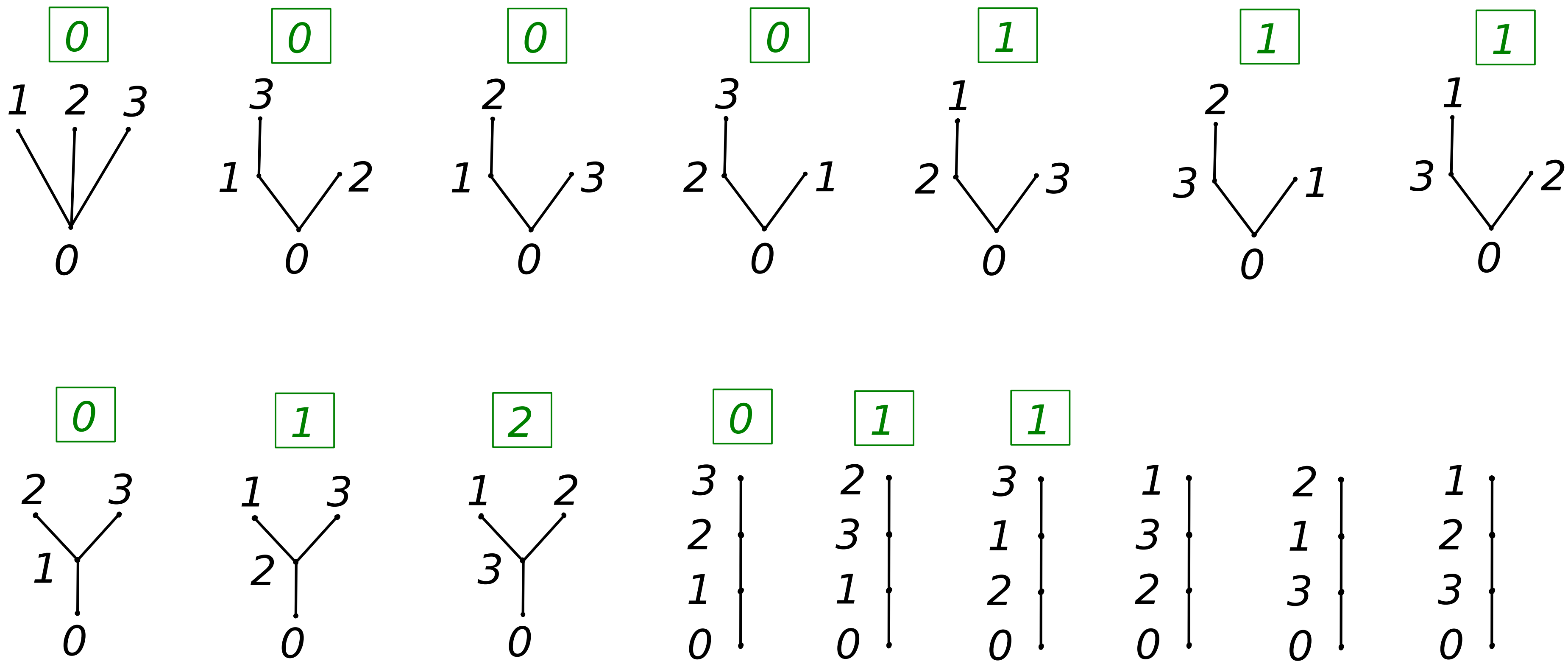
Inversions



Inversions

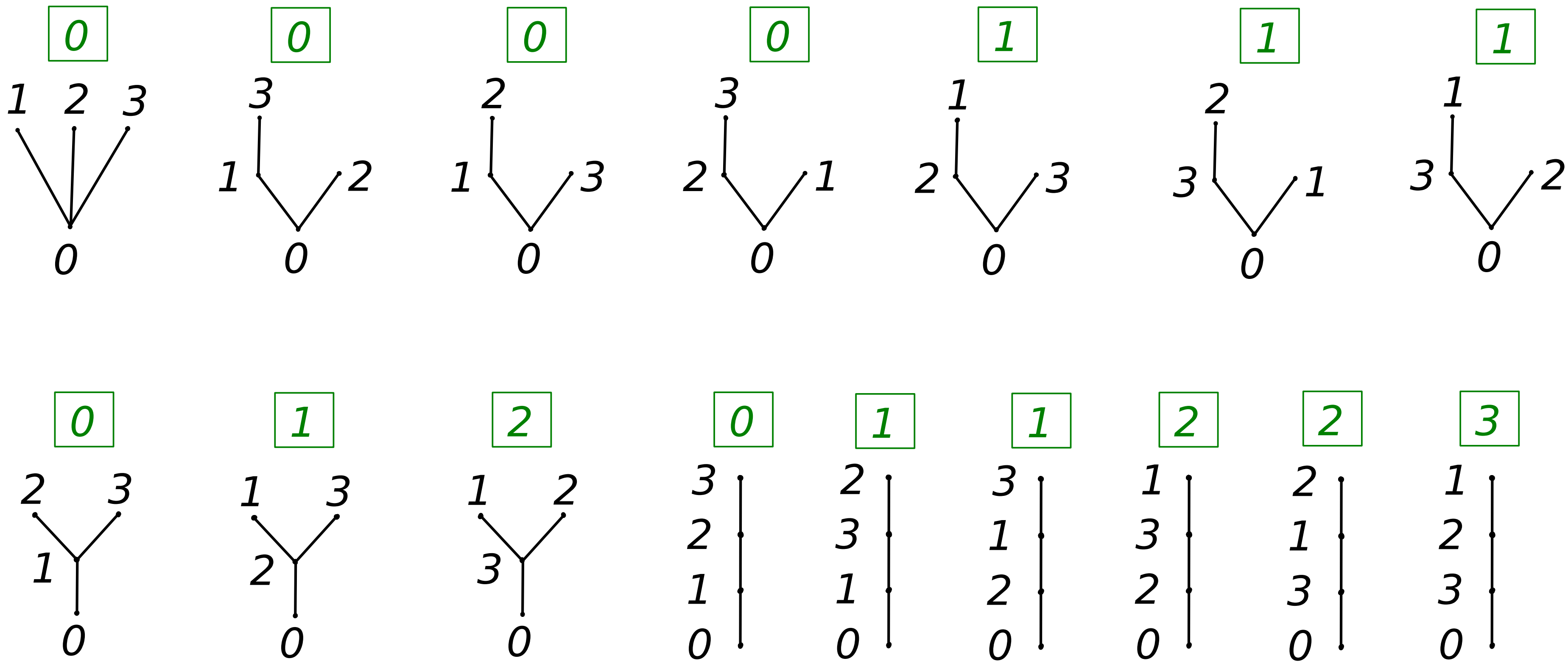


Inversions



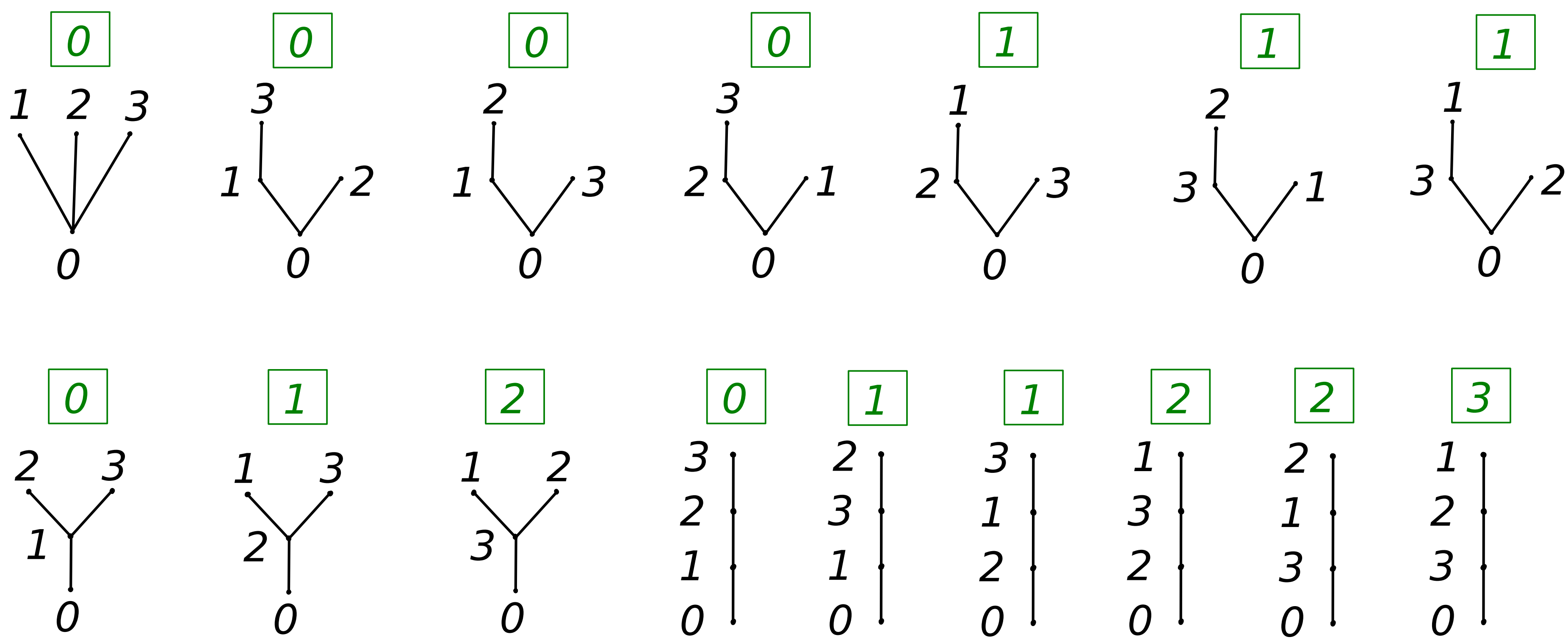
0 1 2 3 0
0 3 1 2 0
0 2 1 3 0
0 3 2 1 0
1 1 2 3 0
1 2 3 1 0
1 1 3 2 0
0 2 3 1 0
1 1 3 2 0
2 1 2 3 0
0 3 2 1 0
1 2 3 1 0
1 3 1 2 0
1 3 2 0
2 1 3 0
1 2 3 0

Inversions



Extra: Inversion enumerator

$$I_3(t) = \sum_T t^{\#\mathrm{inv}(T)} = 6 + 6t + 3t^2 + t^3$$



Extra: Inversion enumerator

I.

$$t^n \cdot I_n(t+1) = \sum_{\substack{G \text{ connected} \\ \text{w/ } n+1 \\ \text{labeled vertices}}} t^{\#\,\text{edges}(G)}$$

Extra: Inversion enumerator

I.

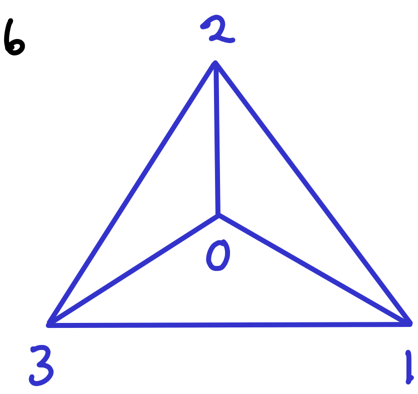
$$t^n \cdot I_n(t+1) = \sum_{\substack{G \text{ connected} \\ \text{w/ } n+1 \\ \text{labeled vertices}}} t^{\#\text{edges}(G)}$$

Example:

$$t^3 I_3(t+1) = t^3\left(6 + 6(t+1) + 3(t+1)^2 + (t+1)^3\right)$$

$$= 16t^3 + 15t^4 + 6t^5 + t^6$$

K_4



Extra: Inversion enumerator

II.

$$\frac{\sum_{n\geq 0} t^{\binom{n+1}{2}} \frac{x^n}{n!}}{\sum_{n\geq 0} t^{\binom{n}{2}} \frac{x^n}{n!}} = \sum_{n\geq 1} I_n(t)(t-1)^n \frac{x^n}{n!}$$

Parking functions

cars

$C_4 \quad C_3 \quad C_2 \quad C_1 \quad \rightsquigarrow$

parking spaces

$\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$

$0 \quad 1 \quad 2 \quad 3$

parking preferences:

$p = p_1 \cdots p_n$

$p_i =$ preferred spot for C_i

Parking functions

cars

parking spaces

$C_4 \quad C_3 \quad C_2 \quad C_1 \quad \rightsquigarrow \quad \underset{0}{\underline{C_2}} \quad \underset{1}{\underline{C_1}} \quad \underset{2}{\underline{C_4}} \quad \underset{3}{\underline{C_3}}$

$$p = p_1 p_2 p_3 p_4 = 1032$$

Parking functions

cars

parking spaces

$c_4 \quad c_3 \quad c_2 \quad c_1 \quad \rightsquigarrow \quad \underline{c_1} \quad \underline{c_2} \quad \underline{c_3} \quad \underline{c_4}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \quad\;\; 1 \quad\;\; 2 \quad\;\; 3$

$p = 0\,1\,0\,0$

Parking functions

cars

$C_4 \quad C_3 \quad C_2 \quad C_1 \quad \rightsquigarrow$

parking spaces

C_1		C_2	C_3
0	1	2	3

$p = 0222$

Parking functions

parking functions

$$1032 \rightsquigarrow \begin{array}{cccc} C_2 & C_1 & C_4 & C_3 \\ \hline 0 & 1 & 2 & 3 \end{array}$$

$$0100 \rightsquigarrow \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \\ \hline 0 & 1 & 2 & 3 \end{array}$$

non-parking function

$$0222 \rightsquigarrow \begin{array}{cccc} C_1 & & C_2 & C_3 \\ \hline 0 & 1 & 2 & 3 \end{array}$$

Parking functions

non-parking function $\Big\}$ 0 2 2 2 $\rightsquigarrow$

C_1		C_2	C_3
0	1	2	3

For a parking function $p = p_1 p_2 p_3 p_4$:

*	At	most	1	car(s)	can	prefer	spot	≥ 3
*	"	"	2	"	"	"	"	≥ 2
*	"	"	3	"	"	"	"	≥ 1
*	"	"	4	"	"	"	"	≥ 0

Parking functions

Proposition. $p = p_1 \cdots p_n$ is a parking function iff for $i = 0, \ldots, n-1$,

$$\#\text{ cars preferring } \geq n-i \text{ is at most } i.$$

Parking functions

Proposition. $p = p_1 \cdots p_n$ is a parking function iff for $i = 0, \ldots, n-1$,

$$\#\text{ cars preferring } \geq n-i \text{ is at most } i.$$

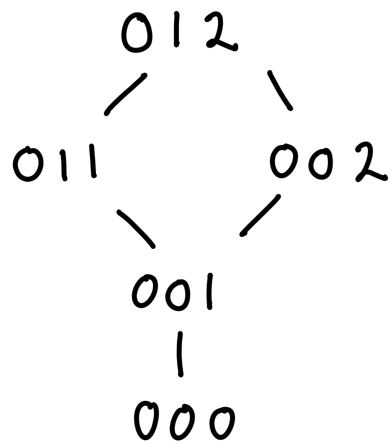
Corollary. Let $p = p_1 \cdots p_n$ be a parking function. Then

(i) S_0 is any permutation of the p_i.

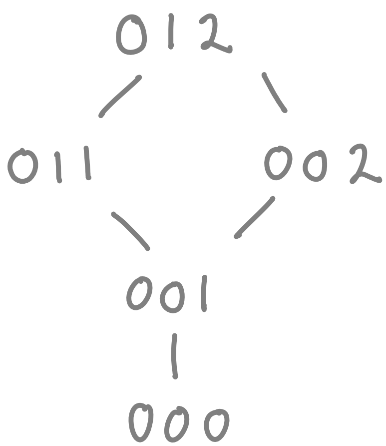
(ii) S_0 is q if $0 \leq q_i \leq p_i \quad \forall i$.

All parking functions: $n = 3$

sorted:



All parking functions: $n = 3$



012 021 102 120 201 210

011 101 110 002 020 200

001 010 100

000

total number $= 16 = (n+1)^{n-1}$

All parking functions: $n = 3$

						#	$\deg(p) = \sum_i p_i$
012	021	102	120	201	210	6	3
011	101	110	002	020	200	6	2
001	010	100				3	1
000						1	0

Tree inversions and parking functions

	#	deg(p)	# inv(T)
012 021 102 120 201 210	6	3	0
011 101 110 002 020 200	6	2	1
001 010 100	3	1	2
000	1	0	3

						#	$\deg(p)$	$\# \mathrm{inv}(T)$
012	021	102	120	201	210	6	3	0
011	101	110	002	020	200	6	2	1
001	010	100				3	1	2
000						1	0	3

Theorem (Kreweras, 1980). Let $g = \mathrm{genus}(K_{n+1}) = \frac{n(n-1)}{2}$. Then

$$I_n(t) = \sum_{p \in PF_n} t^{\,g - \deg(p)}.$$

Theorem (Kreweras, 1980). $g = \text{genus}(K_{n+1}) = \frac{n(n-1)}{2}$.

coefficient of t^i in $I_n(t)$ = # p.f.s of degree $g-i$.

R. Stanley, "An Intro. to Hyperplane Arrangements"

Chapter 6, exercise 4: Find a bijection,

$$\{\text{labeled trees on } 0, \ldots, n\} \longleftrightarrow \{\text{p.f.s } p_1 \cdots p_n\}$$

$$T \longleftrightarrow P_T$$

s.t. $\#\,\text{inv}(T) = g - \deg(P_T)$.

	#	deg(p)	# inv(T)
012 021 102 120 201 210	6	3	0
011 101 110 002 020 200	6	2	1
001 010 100	3	1	2
000	1	0	3

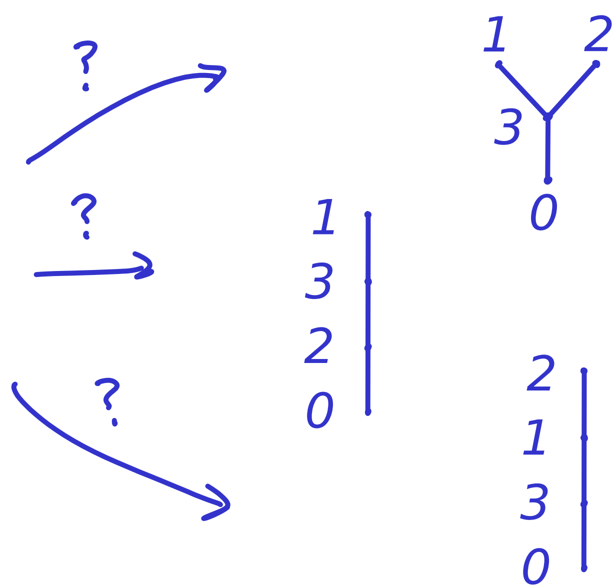
000 $\leftrightsquigarrow$ 1 2 3 0

deg = 0

#inv = 3

						#	deg(p)	#inv(T)
012	021	102	120	201	210	6	3	0
011	101	110	002	020	200	6	2	1
	001	010	100			3	1	2
		000				1	0	3

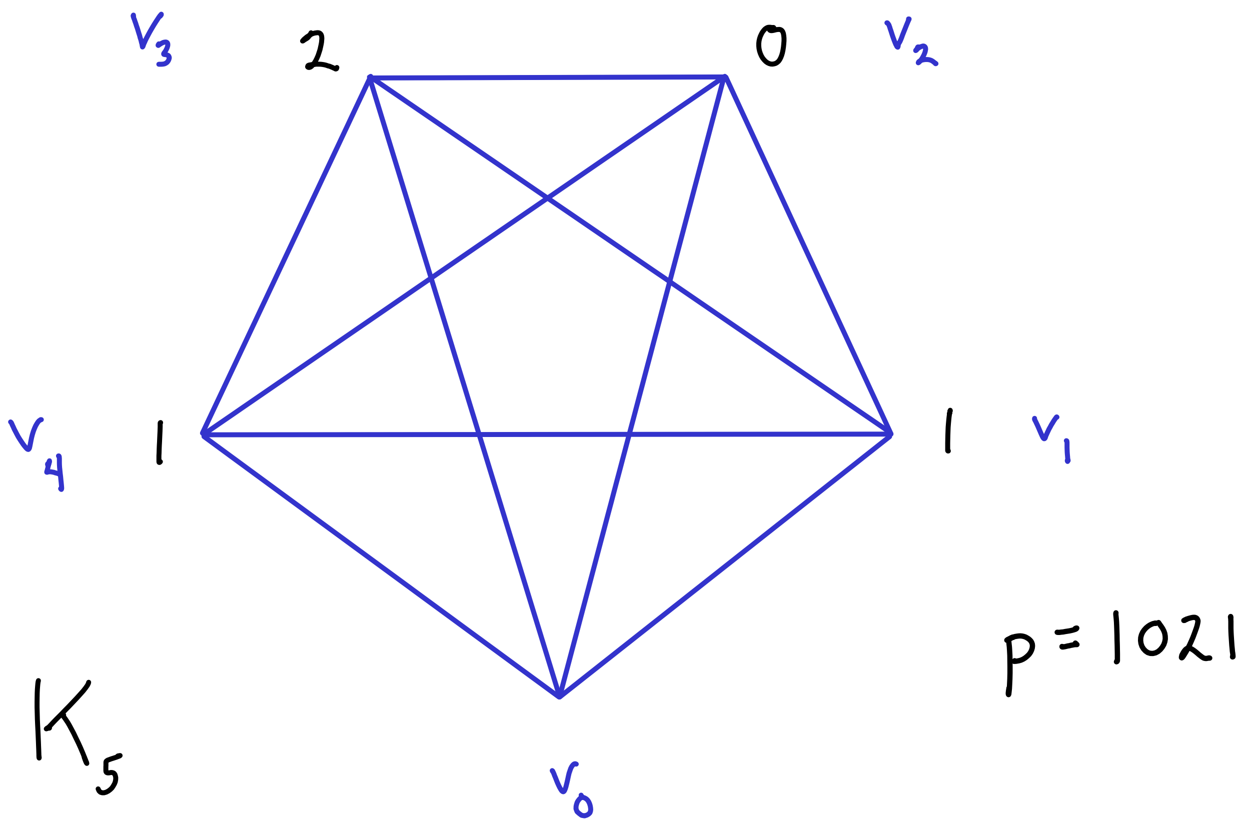
deg = 1 001



#inv = 2

Solution (P., Qiaoyu Yang, Kuai Yu, 2013)

Dhar's algorithm with depth-first search



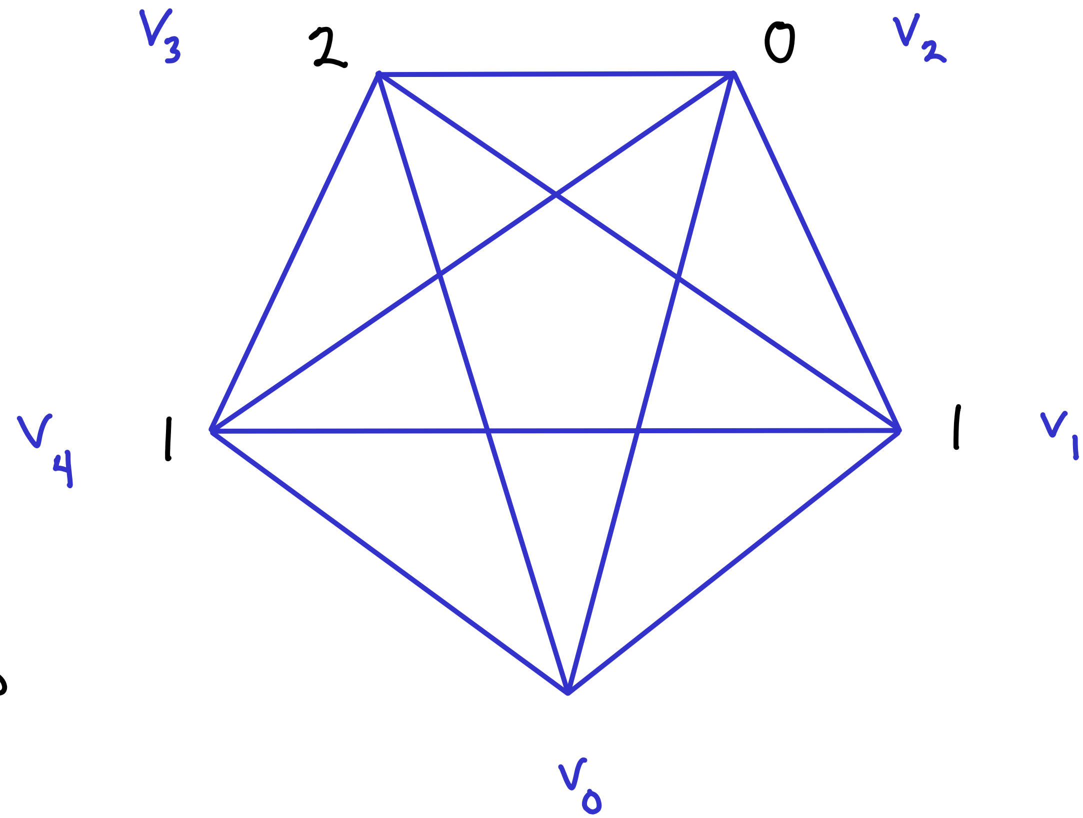
Dhar's algorithm with depth-first search

$p = 1021$

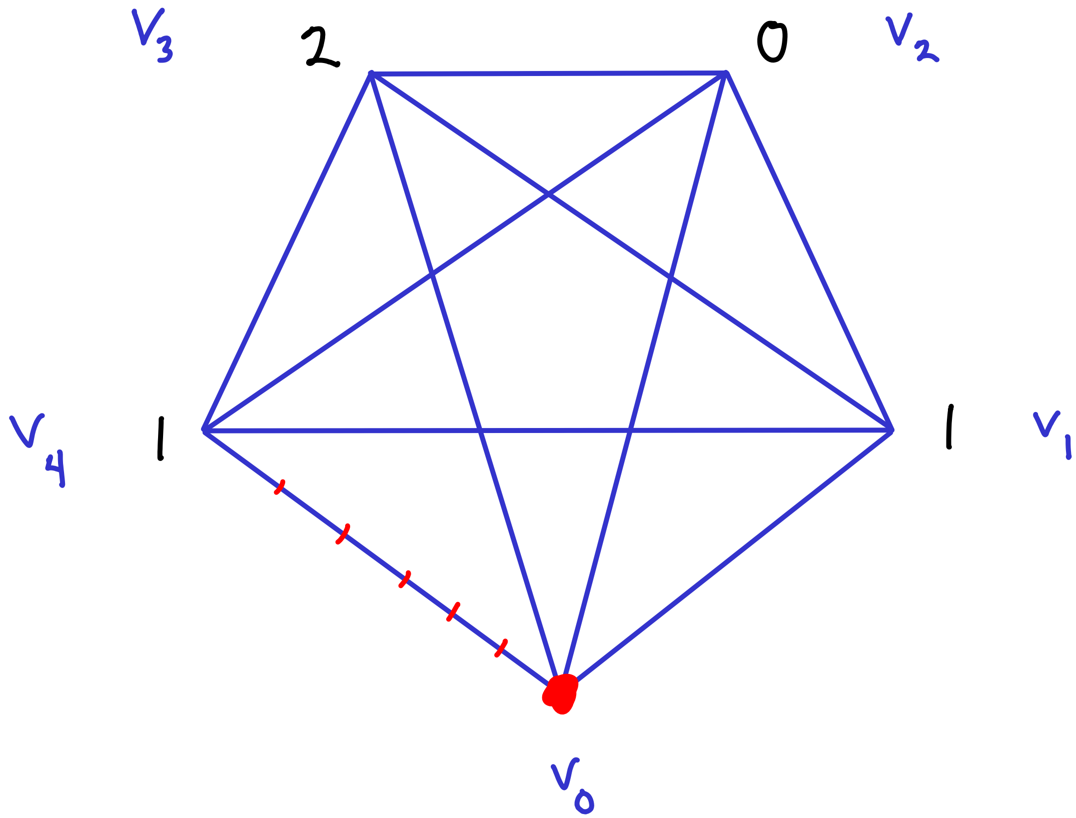
* p_i = # firefighters on vertex v_i, $i = 1, \ldots, n$

* ignite vertex v_0

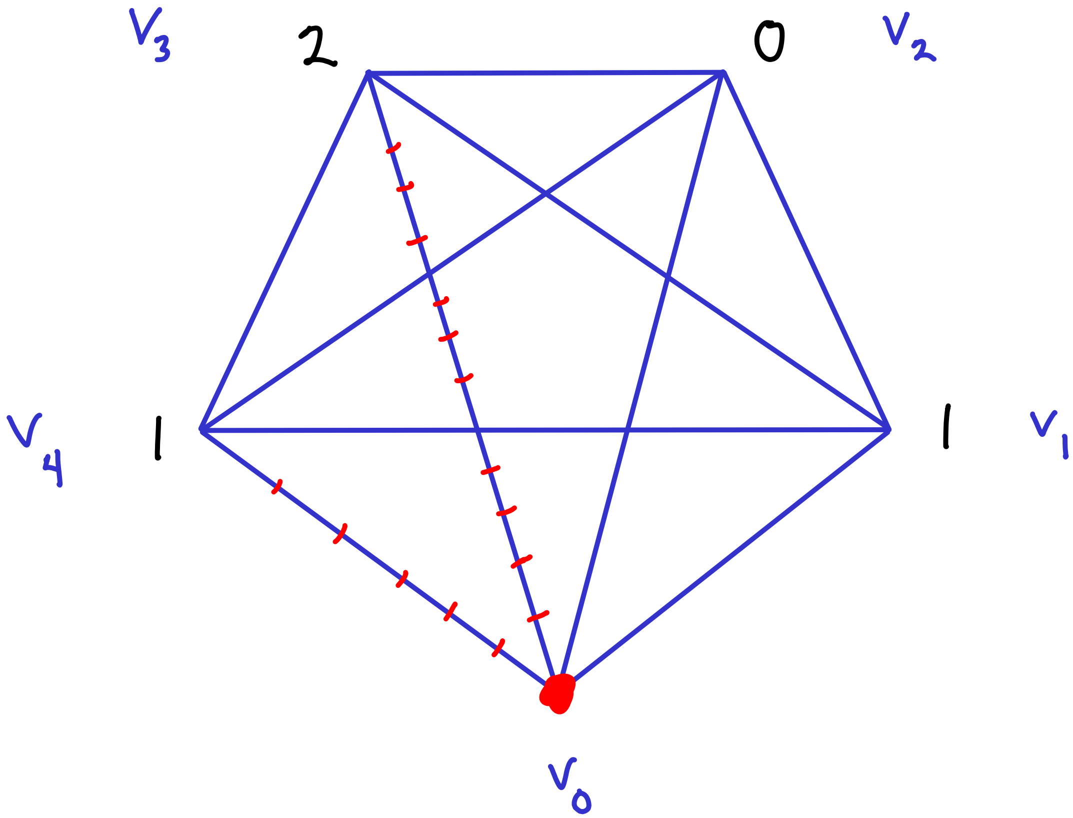
* fire disperses according to a depth first search



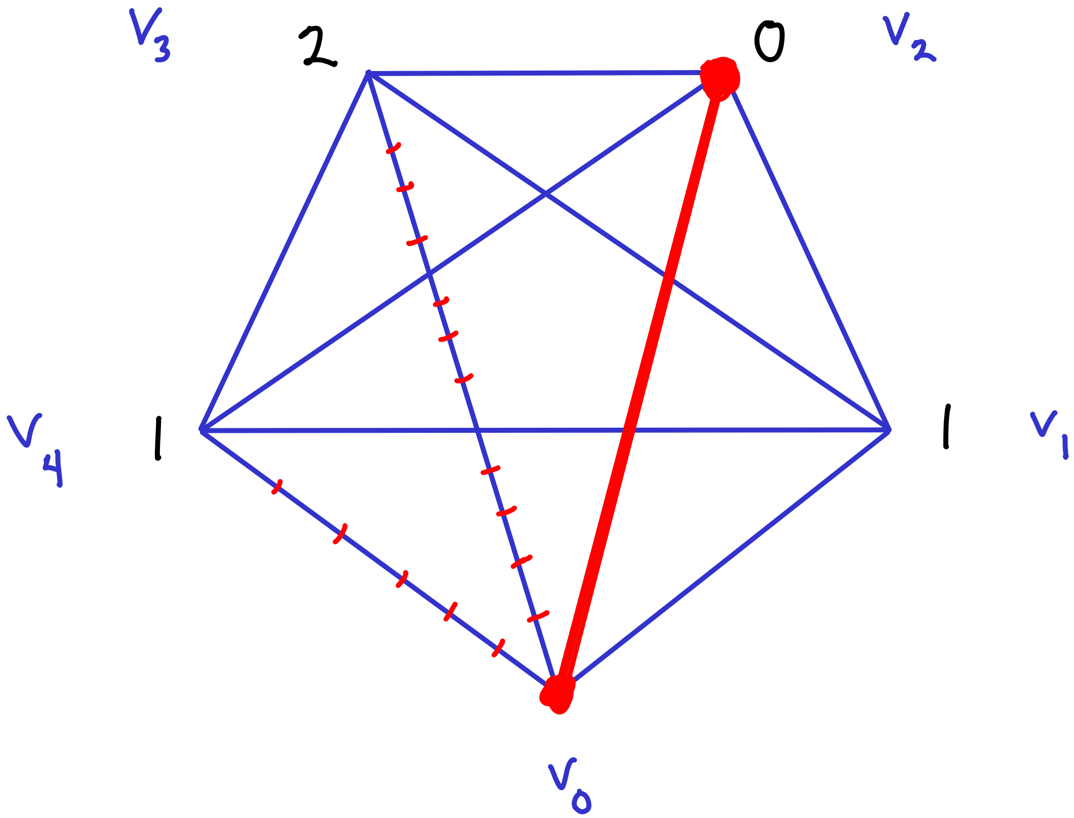
Dhar's algorithm with depth-first search



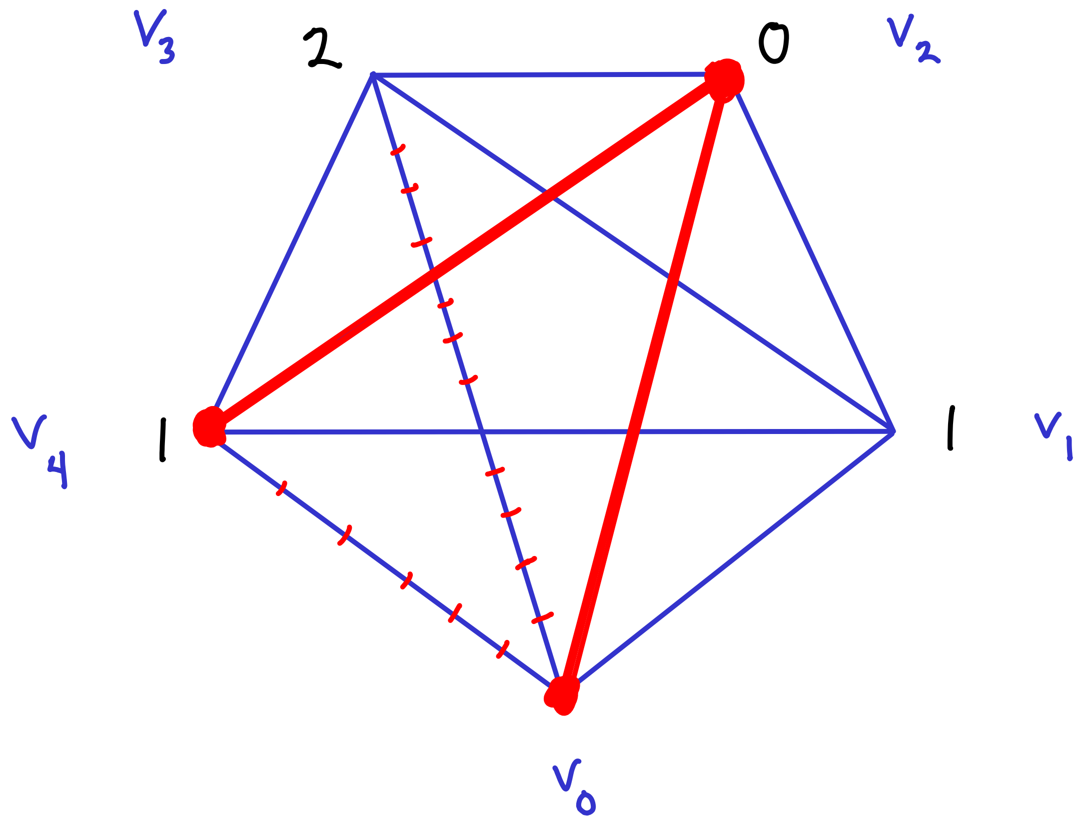
Dhar's algorithm with depth-first search



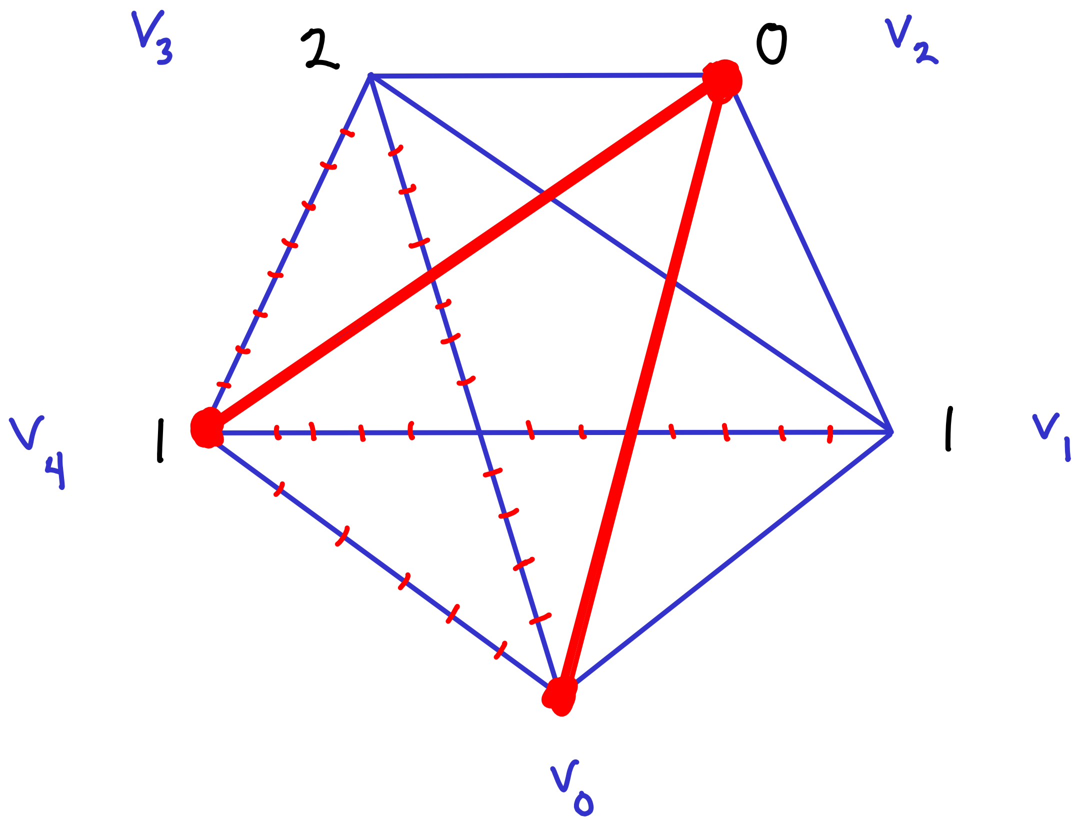
Dhar's algorithm with depth-first search



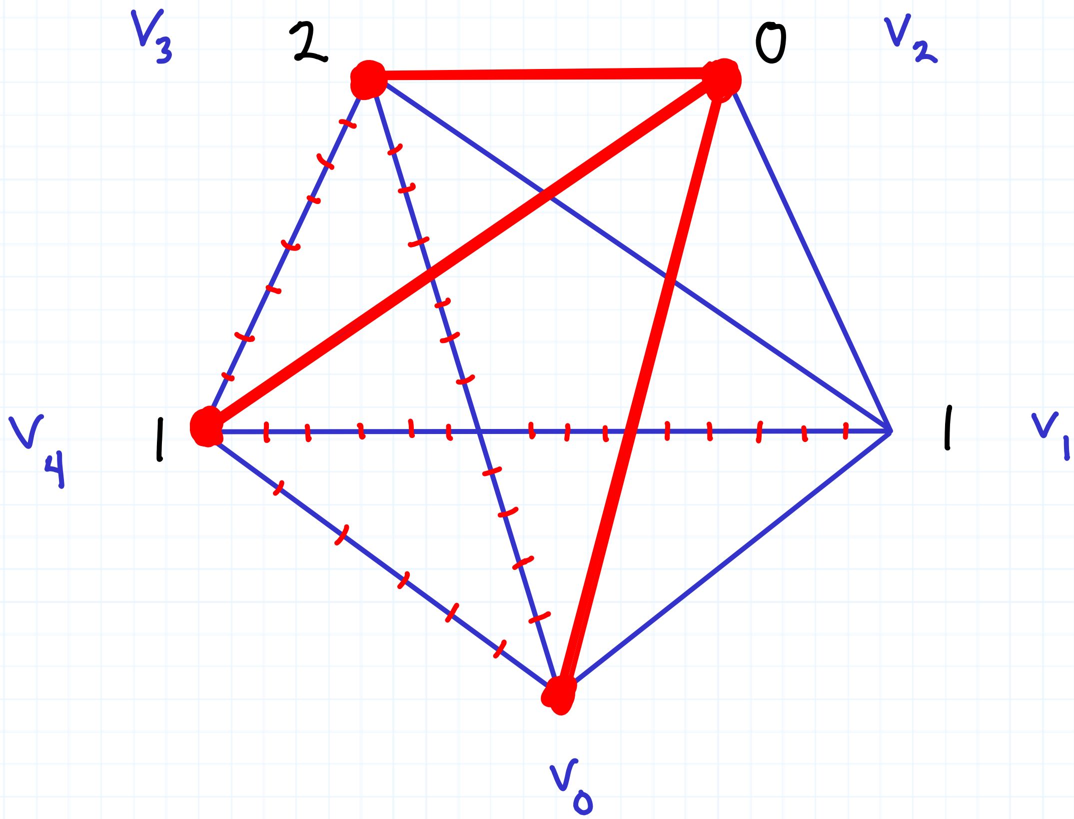
Dhar's algorithm with depth-first search



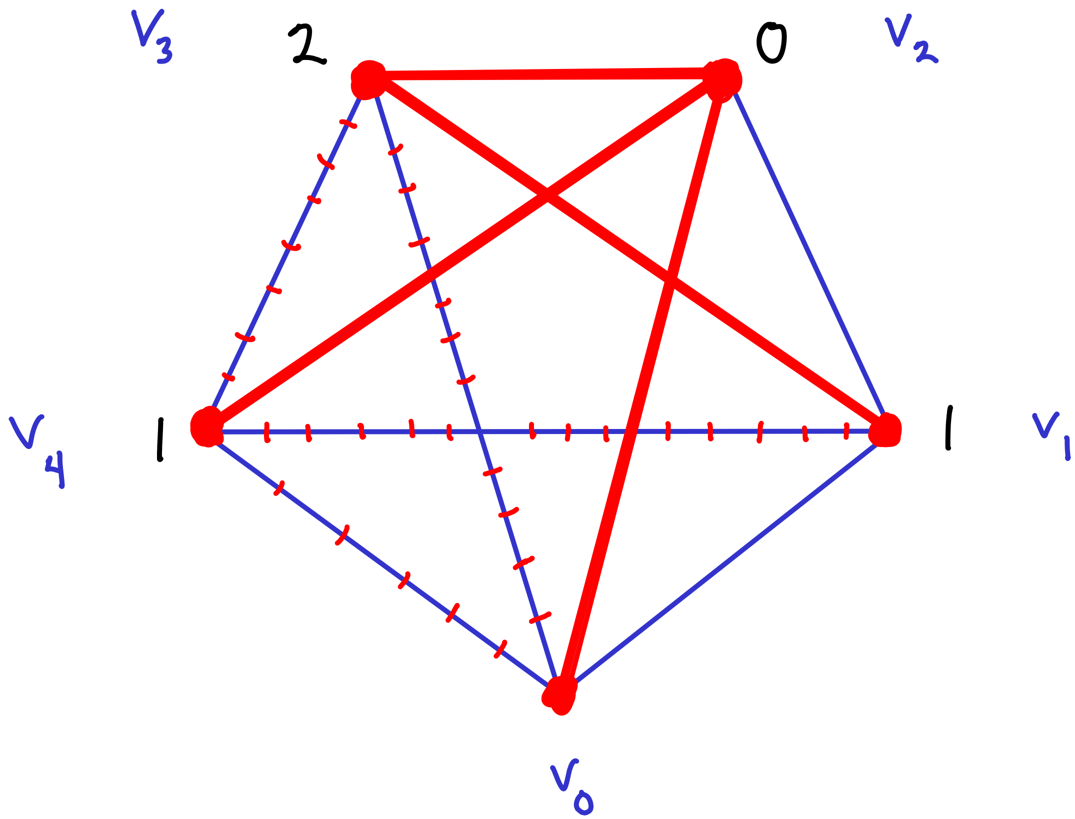
Dhar's algorithm with depth-first search



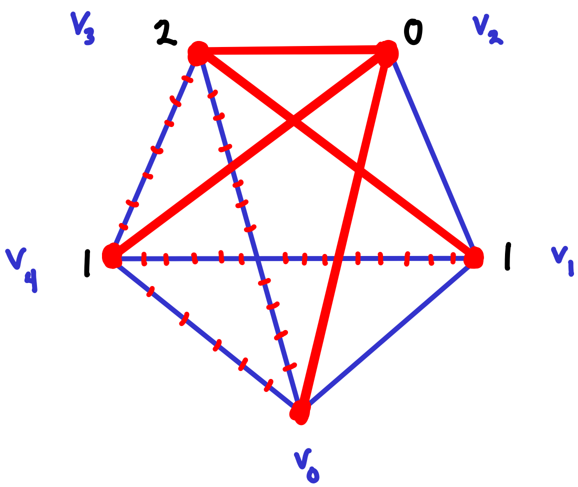
Dhar's algorithm with depth-first search



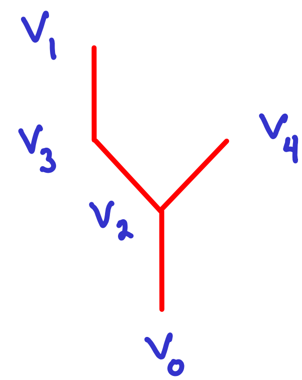
Dhar's algorithm with depth-first search



Dhar's algorithm with depth-first search



$p = 1021 \rightsquigarrow$



$$g(K_5) = \frac{4 \cdot 3}{2} = 6$$

$$g - \deg(p) = 6 - 4 = \# \text{inv} = 2$$

Generalization

Theorem (PYY, 2013). Let G be a simple connected graph. Then $\exists$ bijection

$$\{\text{spanning trees of } G\} \longleftrightarrow \{G\text{-parking functions}\}$$

$$T \longleftrightarrow P_T$$

$$\#\,K\text{-inv}(T) = \text{genus}(G) - \deg(P_T)$$

$\text{genus}(G) = \#\text{edges} - \#\text{vertices} + 1.$

Thanks!

References

* G-parking functions and tree inversions, Perkinson, Yang, Yu arXiv, 2013. To appear in Combinatorica.

* A new bijection between forests and parking functions, H. Shin arXiv, 2008.