Math 372 lecture for Monday, Week 14

**Firing posets IV.**

Today's lecture is based on results in Characterisation of Lattices Induced by (extended) Chip Firing Games by Magnien, Phan, and Vuillon.

Let $\mathcal{FL}$ be the set of firing lattices obtained from divisors on finite directed multigraphs. Then the following hold:

1. $\mathcal{FL}$ contains every finite distributive lattice.
2. Every firing lattice is locally free.
3. Every firing lattice is equivalent to a *simple* firing lattice, i.e., a firing lattice in which each vertex in the underlying graphs fires at most once.
4. There exist finite locally free lattices which are not isomorphic any firing lattice.

**Example.** Figure 1 is a locally free lattice that is not isomorphic to any firing lattice. (The letters are using in the proof.) It is easy to check the lattice is locally free. Suppose it

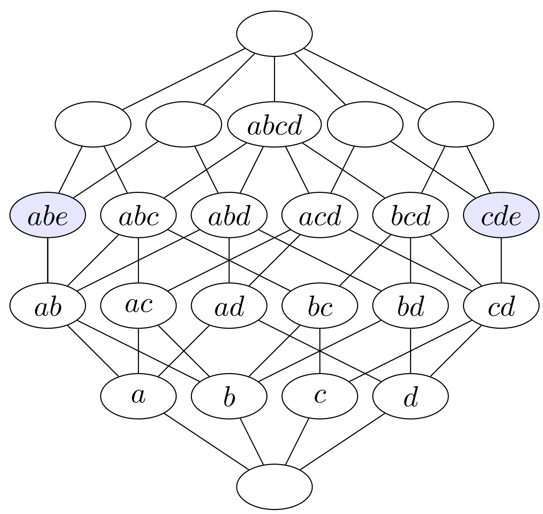



Figure 1: A locally free lattice that is not a firing lattice.

were a firing lattice. Then we may assume it is a simple firing lattice (in which each vertex fires at least once). The labels represent sequences of vertices that are fired to reach that lattice element. There is a sublattice which is the Boolean lattice $B$ of the subsets of the vertices $a, b, c, d$. The node $ab$ has a cover that leave that Boolean lattice. So there must be a vertex $e$ that is fired to get to the divisor representing that cover. Similarly, there is a vertex $f$ that is fired to get to the cover of $cd$ that is not in $B$. However, it must be that

$f = e$. To see that, try filling in the rest of the labels. The maximal lattice element must be labeled by a string consisting of 5 letters representing a path from the minimal element to the maximal element, and that label is independent of the path. (If $D \xrightarrow{\sigma} D'$ and $D \xrightarrow{\tau} D'$ and the supports of $\sigma$ and $\tau$ do not contain every vertex, than $\sigma = \tau$.)

Now note that vertex $e$, since first becomes unstable after $a$ and $b$ fire or after $c$ and $d$ fires. Let $n$ be the number of chips needed to fire $e$. Let $n_a, n_b, n_c, n_d$ be the number of edges connecting $a, b, c, d$, respectively, to $e$. We may assume $n_a \geq n_b$ and $n_c \geq n_d$. Then

$$n \leq n_a + n_b \leq 2n_a \quad \text{and} \quad n \leq n_c + n_d \leq 2n_c.$$

Adding the inequalities gives

$$2n \leq 2n_a + 2n_c.$$

However, then $n \leq n_a + n_c$, and $e$ would come unstable after firing $a$ and $c$. So the node labeled $ac$ should be covered by a node $ace$, and it is not.

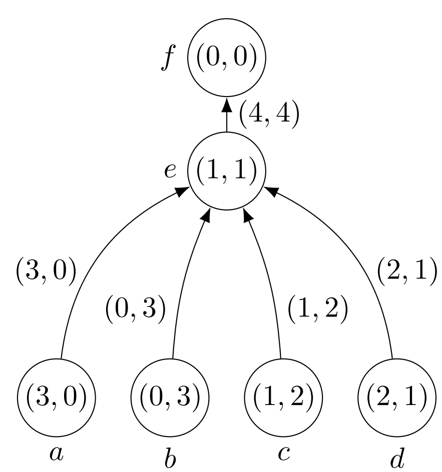
**Chromatic chip-firing.** Here we consider a chip-firing game that involves chips of various colors on a multigraph of edges of various colors. Let $C = [n]$ be a finite collection of colors, and let $G = (V, E)$ be a graph with a finite set of vertices $V$ and edges $E$. Let $\gamma\colon E \to \mathbb{N}^C \approx \mathbb{N}^n$. If $\gamma(e) = (c_1, \ldots, c_n)$, we think of $e$ as being a multiset of $\sum_{i=1}^n c_i$ edges, with $c_i$ edges of color $i$ for each $i$. A *(chromatic) divisor* on $G$ is an assignment $\chi\colon V \to \mathbb{N}^C$ of colored chips to each vertex. To fire a vertex from a divisor $\chi$, we replace $\chi$ with the divisor $\chi'$ where

$$\chi'(w) = \begin{cases} \chi(v) - \sum_{vu \in E} \gamma(vu) & \text{if } w = v \\ \chi(w) + \gamma(vw) & \text{if } vw \in E \\ \chi(w) & \text{otherwise.} \end{cases}$$

Firing the vertex $v$ is *legal* if $\chi(v) \geq \sum_{vu \in E} \gamma(vu)$.

We may then associate a firing graph to each divisor, as before.

**Example.** (Due to Zach.) Here is a chromatic divisor whose firing graph is isomorphic to the locally-free lattice in Figure 1:



**Problems.**

1. Is every firing lattice isomorphic to a firing lattice for a divisor on an *undirected* graph?
2. Extend this work to finitary graphs and finitary lattices.
3. Study the colored chip firing game.