Math 372 lecture for Wednesday, Week 14

Firing posets II.

Last time, we discussed the following two results:

Theorem 1. (Fundamental theorem for finitary distributive lattices) Let P be a poset in which every principal order ideal is finite. Then the poset $J_f(P)$ of *finite* order ideals of P, ordered by inclusion, is a finitary distributive lattice. Conversely, if L is a finitary distributive lattice and P is its subposet of join-irreducibles, then every principal order ideal of P is finite, and $L \simeq J_f(P)$.

Theorem 2. Let P be an ideal in which every principal order ideal is finite. Define a poset $\widetilde{P}$ by adding elements α and β to P, then defining

$$\alpha < \beta < x$$

for all $x \in P$. Let $G = G(P)$ be the Hasse diagram for $\widetilde{P}$. For each $x \in P$, let $n(x)$ be the number of covers of x in P. Define $D \in \mathrm{Div}(G)$ by

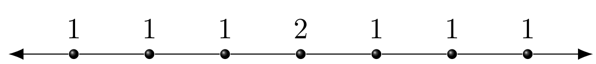
$$D(x) = \begin{cases} 0 & \text{if } x \in \{\alpha, \beta\}, \\ \deg_G(x) & \text{if } x \text{ is a minimal element of } P, \\ n(x) & \text{otherwise.} \end{cases}$$

Then the firing graph $\mathcal{F}(D)$ has no cycles, and when considered as a poset, $\mathcal{F}(D) \simeq J_f(P)$.

Combining these, we get the immediate corollary:

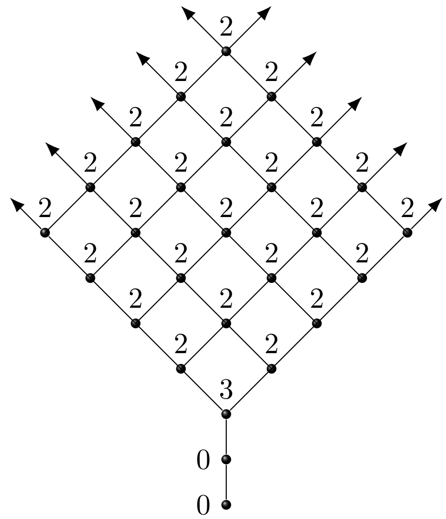
Corollary. Every finitary distributive lattice is the lattice associated with the firing graph of a divisor on some graph.

Example. We saw previously that Young's lattice is given by the firing lattice of the divisor:



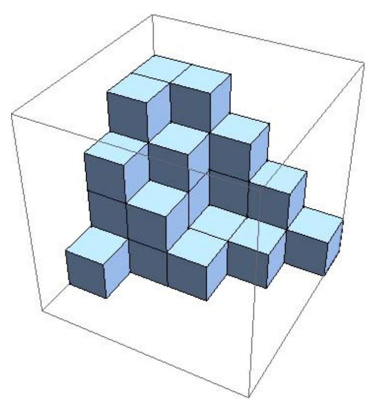
However, this is not the divisor that would be given by Theorem 2. To see this, first note that the join irreducibles of Young's lattice correspond to the Young diagrams that are (m, n) rectangles for any $m, n \in \mathbb{P} := \{1, 2, \dots\}$. (We use $\mathcal{P}$ here rather than $\mathbb{N}$ since the minimal element of a poset is not join irreducible.) These are the only Young diagrams from which there is exactly one choice for a rectangle to remove and still have a Young diagram. Thus, according to Theorem 1, Young's diagram is isomorphic to $J_f(\mathbb{P} \times \mathbb{P}) \simeq J_f(\mathbb{N} \times \mathbb{N})$.

Applying Theorem 2 to $\mathbb{N} \times \mathbb{N}$, we get the much more complicated divisor whose firing graph also gives Young's diagram:



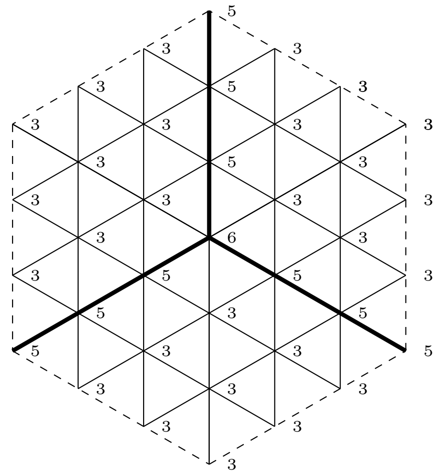
Question. Is there some way to look at the above divisor and infer our earlier, simpler, solution?

Example. Young's lattice belongs the family of lattices $J_f(\mathbb{N}^k)$ for $k = 1, 2, \ldots$ The case $k = 3$ is known as the lattice of plane partitions. Just as Young's lattice can be thought of as shoving squares into a corner, plane partitions correspond to pushing 3-dimensional boxes into the corner of a room:



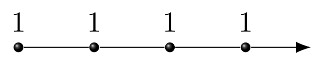
A plane partition

The plane partition lattice comes from the firing graph of the following divisor:

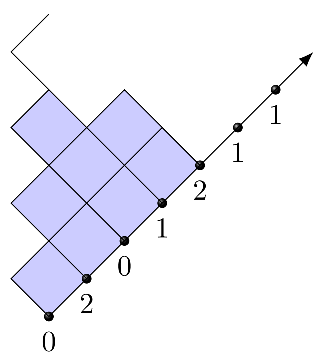


There is also a more complicated divisor arising through an application of Theorem 2.

Example. Consider the firing graph of the following divisor:



The corresponding lattice is the lattice of "shifted shapes". "A shifted shape" is a Young diagram for which the corresponding partition has unequal parts. For instance, the shifted shape for the partition $(4, 3, 1)$ can be written like this



What does the poset of join irreducibles look like for the lattice of shifted shapes?

We have seen that every finitary distributive lattice is the lattice associated to a firing graph for the divisor of some graph.

Main question: Can we characterize the posets arising from firing graphs?

We will focus on this question now.

A *firing script* is a formal sum of vertices: $\sigma = \sum_{i=1}^{n} a_i v_i$. It is *legal* for a divisor D if there is a legal sequence of vertex firings $w_1, \ldots, w_k$ where $k = \sum_{i=1}^{n} a_i$ such that $\sigma = \sum_{i=1}^{k} w_i$.

A lattice is *locally free* if for each element x, the interval from x to the join of all elements covering x is a Boolean lattice (i.e., isomorphic to a lattice of subsets of a given set).

Theorem. (Björner, Lovàsz, Shor, 1991). Let G be a finite graph, and let $D \in \mathrm{Div}(G)$ be stabilizable. Then the firing graph $\mathcal{F}(D)$ is a locally free lattice. Given $D', D'' \in \mathcal{F}(D)$, let σ' and σ'' be legal firing scripts such that $D \xrightarrow{\sigma'} D'$ and $D \xrightarrow{\sigma''} D''$. Then $D \xrightarrow{\sigma' \vee \sigma''} D''$, where $(\sigma' \vee \sigma'')(v) := \max\{\sigma'(v), \sigma''(v)\}$ for all vertices v of G.

In our case, given $D' \in \mathcal{F}(D)$ (with D stabilizable), we look at all the vertices of D' that are unstable. Say these are $v_1, \ldots, v_k$. Suppose $D \xrightarrow{\sigma'} D'$ via a legal firing script σ'. Then let $\tau = \sigma' + v_1 + \cdots + v_k$. The join discussed above is $D'' := D - L\tau$, so $D \xrightarrow{\tau} D''$.