Math 372 lecture for Monday, Week 13

Distributive lattices

Let P be a poset. A *greatest lower bound* for $x, y \in P$ is an element $v \in P$ that is a lower bound, i.e., $v \leq x$ and $v \leq y$, and such that for all $w \in P$

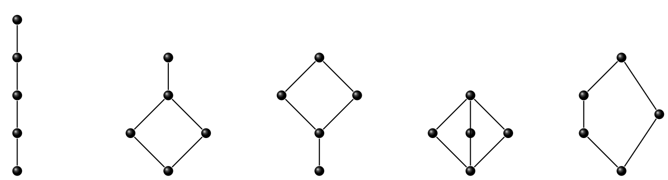
$$w \leq x \text{ and } w \leq y \quad \Longrightarrow w \leq v.$$

A *least upper bound* for elements $x, y \in P$ is an element $u \in P$ that is an upper bound, i.e., $x \leq u$ and $y \leq u$, and such that for all $w \in P$,

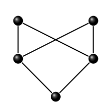
$$x \leq w \text{ and } y \leq w \quad \Longrightarrow u \leq w.$$

A greatest lower bound for x and y, if it exists, is unique. It is called the *meet* of x and y and denoted $x \wedge y$ A least upper bound for x and y, if it exists, is unique. It is called the *join* of x and y and denoted $x \vee y$.

A *lattice* is a poset L in which every pair of elements $x, y \in L$ as a meet and a join. Here are Hasse diagrams for all lattices with five elements:



Here is a poset that is not a lattice:



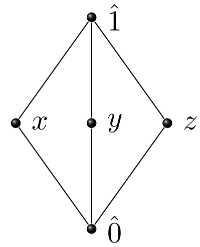
Distributive lattices. Let L be a lattice. Then L is *distributive* if for all $x, y, z \in L$,

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

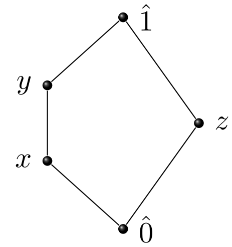
One may show that this condition is equivalent to the condition that

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Examples of distributive lattices include the Boolean posets B_n, the nonnegative integers $\mathbb{N}$, and Young's lattice (of partitions of integers). Here are two examples of lattices that are not distributive:



Diamond lattice, M_3



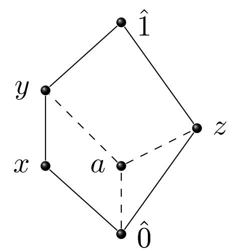
Pentagon lattice, N_5

These examples characterize non-distributive lattices.

Proposition. A lattice is distributive if and only if none of its sublattices is isomorphic to M_3 or N_5.

A *sublattice* of a lattice is a subset that is closed under the meet and join operations of the original lattice.

Example. Here is an example of a lattice that contains N_5 as a subset by not a sublattice:



A distributive lattice.

A lattice is *finitary* if it is locally finite (meaning that each of its intervals is finite) and if it has a unique smallest element $\widehat{0}$.

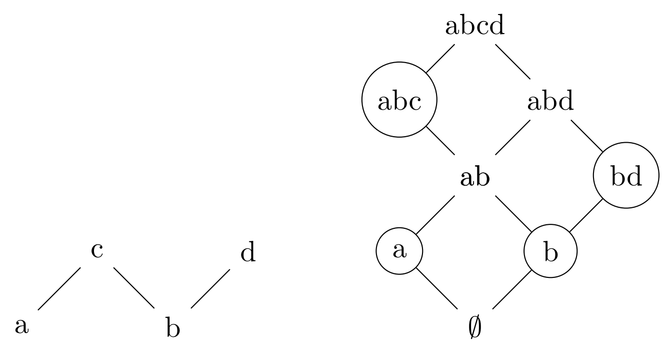
Given any poset P, we now describe a way of constructing an distributive lattice. An *order ideal* or *down-set* in P is a subset I of P such that if $x \in I$ and $y \leq x$, then $y \in P$. A *principal order ideal* is an order ideal generated by a single element, i.e., of the form $\{y : y \leq x\}$ for some $x \in P$. For each $x \in P$ we denote the corresponding principal order ideal by

$$\Lambda_x := \{y \in P : y \leq x\}\,.$$

Let $J(P)$ be the set of all order ideals of P and give it a poset structure by inclusion of subsets of P. Then $J(P)$ is a lattice in which the meet operation is intersection and the join operation is union (as subsets of P).

An element x in a lattice L is *join irreducible* if $x \neq \widehat{0}$ and if it is not possible to write $x = y \vee z$ with $y < x$ and $z < x$. In other words, x is join irreducible if it covers exactly one element.

Example. Hasse diagrams for a poset P and its lattice of join irreducibles $J(P)$ appear below. Subsets are listed as words, e.g., $\{a, b, d\} = abd$. The join irreducibles in $J(P)$ are circled.



P $\qquad$ $J(P)$

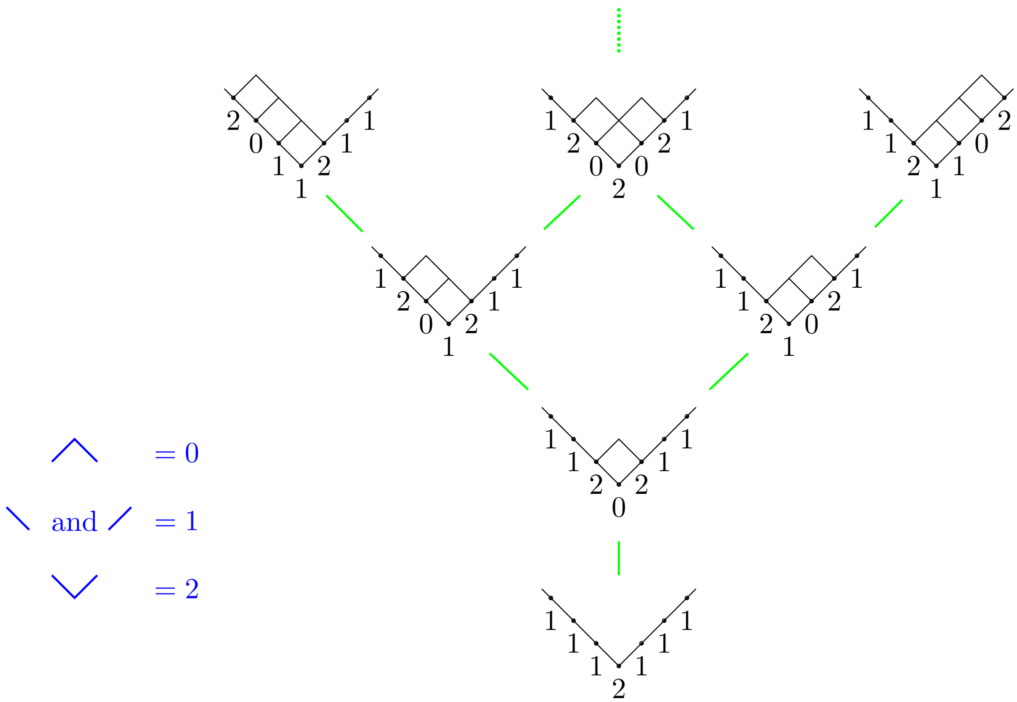
The following theorem is quoted from Stanley's *Enumerative Combinatorics, Volume 1*:

Theorem. (Fundamental theorem for finitary distributive lattices) Let P be a poset in which every principal order ideal is finite. Then the poset $J_f(P)$ of *finite* order ideals of P, ordered by inclusion, is a finitary distributive lattice. Conversely, if L is a finitary distributive lattice and P is its subposet of join-irreducibles, then every principal order ideal of P is finite, and $L \simeq J_f(P)$.

Young's lattice. Let Y be the infinite path graph with vertex set $\mathbb{Z}$ and edges $\{i, i+1\}$ for all $i \in \mathbb{Z}$. Let $D \in \mathrm{Div}(Y)$ be given by

$$D(i) = \begin{cases} 2 & \text{if } i = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Then $\mathcal{F}(D)$ is isomorphic to Young's lattice:



The proof is obtained by looking at the path that runs along the top of each diagram. It extends infinitely in the $(-1, 1)$ and in the $(1, 1)$ directions and has bumps along the top of the Young diagram. The method of decoding the lines and bumps into the numbers 0, 1, and 2 is listed in blue on the left.

Theorem. Let P be a poset in which every principal order ideal is finite. Define a poset $\widetilde{P}$ by adding elements α and β to P, then defining

$$\alpha < \beta < x$$

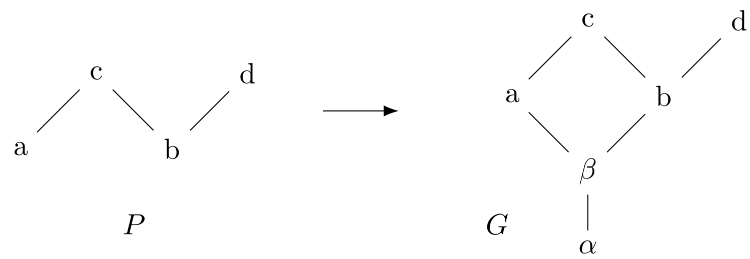
for all $x \in P$. Let $G = G(P)$ be the Hasse diagram for $\widetilde{P}$. For each $x \in P$, let $n(x)$ be the number of covers of x in P. Define $D \in \mathrm{Div}(G)$ by

$$D(x) = \begin{cases} 0 & \text{if } x \in \{\alpha, \beta\}, \\ \deg_G(x) & \text{if } x \text{ is a minimal element of } P, \\ n(x) & \text{otherwise.} \end{cases}$$

Then the firing graph $\mathcal{F}(D)$ has no cycles, and when considered as a poset, $\mathcal{F}(D) \simeq J_f(P)$.

Proof. Homework. $\square$

Example. Apply the construction in the theorem to the zigzag poset example, above:



The divisor described in the theorem is $D = 2a + 2b$, and its firing graph is pictured below: