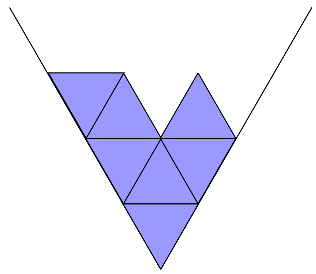
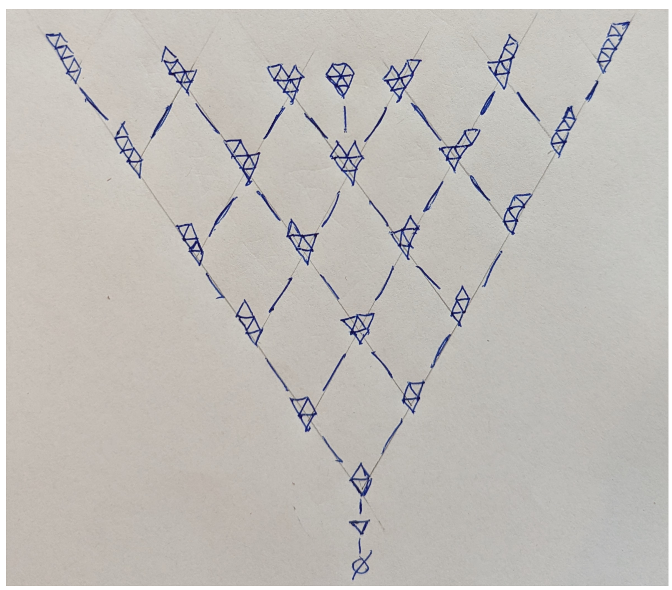
Math 372 lecture for Friday, Week 13

Firing posets III.

Example 1. Consider dropping equilateral triangles into a V-shape:



Analogous to Young's lattice, we get a distributive lattice T:



For homework, you are asked to find its subposet of join irreducibles.

Let $t(n)$ be the number of elements of T having rank n, and let $\tau(x) = \sum_{n\geq 0} t(n)x^n$ be the rank generating function. Here are a few things some students I have shown:

1. The rank generating function is

$$\begin{aligned}\tau(x) &= \prod_{i\geq 1} \frac{(1+x^{2i-1})}{(1-x^{2i})} \\ &= 1 + 1x + 1x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 7x^7 + 10x^8 + 13x^9 + 16x^{10} + 21x^{11} + \cdots\end{aligned}$$

2. The number $t(n)$ is:

- the number of partitions of n in which each even part occurs with even multiplicity and with no restriction on the odd parts;
- the number of partitions of n in which all odd parts occur with multiplicity 1 and with no restriction on the even parts;
- the number of partitions of n into parts not congruent to 2 mod 4.

Question. What is the simplest, most elegant, realization of T as a firing lattice?

Problem. Think of nice variations of these "stacking lattices" and calculate the generating functions.

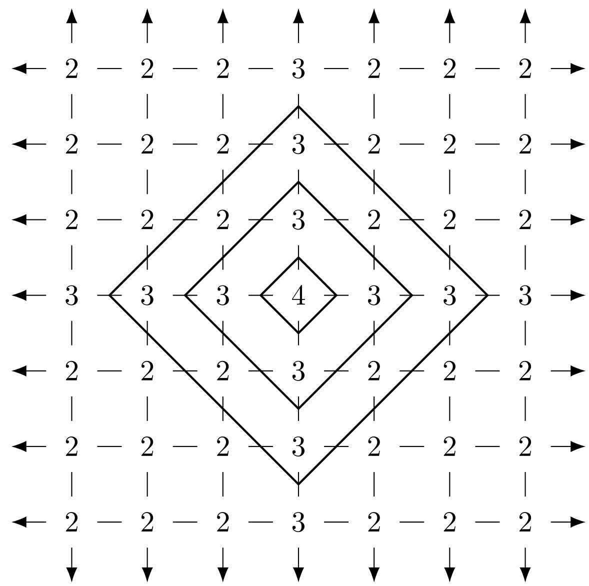
Example 2. Consider the divisor on the integer grid graph pictured below:

$$
\begin{array}{ccccccccccccccc}
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\leftarrow & 2 & - & 2 & - & 2 & - & 3 & - & 2 & - & 2 & - & 2 & \rightarrow \\
 & \mid & & \mid & & \mid & & \mid & & \mid & & \mid & & \mid & \\
\leftarrow & 2 & - & 2 & - & 2 & - & 3 & - & 2 & - & 2 & - & 2 & \rightarrow \\
 & \mid & & \mid & & \mid & & \mid & & \mid & & \mid & & \mid & \\
\leftarrow & 2 & - & 2 & - & 2 & - & 3 & - & 2 & - & 2 & - & 2 & \rightarrow \\
 & \mid & & \mid & & \mid & & \mid & & \mid & & \mid & & \mid & \\
\leftarrow & 3 & - & 3 & - & 3 & - & 4 & - & 3 & - & 3 & - & 3 & \rightarrow \\
 & \mid & & \mid & & \mid & & \mid & & \mid & & \mid & & \mid & \\
\leftarrow & 2 & - & 2 & - & 2 & - & 3 & - & 2 & - & 2 & - & 2 & \rightarrow \\
 & \mid & & \mid & & \mid & & \mid & & \mid & & \mid & & \mid & \\
\leftarrow & 2 & - & 2 & - & 2 & - & 3 & - & 2 & - & 2 & - & 2 & \rightarrow \\
 & \mid & & \mid & & \mid & & \mid & & \mid & & \mid & & \mid & \\
\leftarrow & 2 & - & 2 & - & 2 & - & 3 & - & 2 & - & 2 & - & 2 & \rightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow &
\end{array}
$$

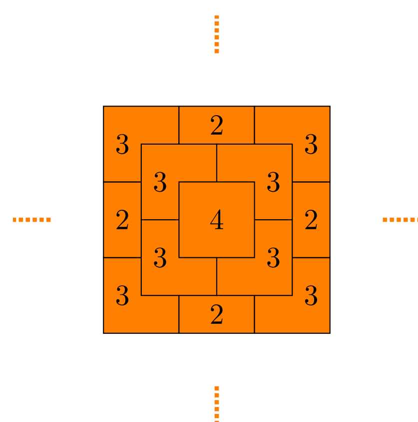
Our goal is to describe the firing lattice $\mathcal{F}(D)$ for this divisor. The minimal element in $\mathcal{F}(D)$ is the divisor itself. It has one unstable vertex, which when fired unstablizes its four neighbors. Then there are four possibilities. So the rank generating function starts out $1, 1, 4$. We will prove the following:

Theorem. Imagine a stack of oranges with levels $1, 2, 3, \dots$, starting at the top with level 1. Level k consists of a $k \times k$ square of k^2 oranges, and it sits on top of the $(k+1) \times (k+1)$-square of oranges on level $k+1$. Let $o(\ell)$ be the number of ways of removing ℓ oranges from the stack so that none fall. (So one may only remove oranges that do not support oranges on a previous level.) Then $\mathcal{F}(D)(x) := \sum_{i \geq 0} o(n)x^n$ is the rank generating function for the divisor D, above.

Proof. Replace the oranges with cubes, and place them on top the divisor graph. So, looking straight down at the stack of cubes, we get a picture like this:



Here is another rendition:



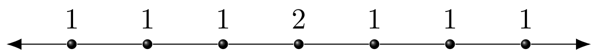
Supposing each box has side length 2, the numbers now refer to the area of the top face of the box that is exposed. Removing a box exposes more area from the boxes below it, and this corresponds exactly to firing vertices. $\square$

Remarks.

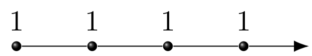
- The same argument given above but in one dimension lower gives an isomorphism from Young's lattice to the firing graph for the divisor $\ldots, 1, 1, 1, 2, 1, 1, 1, \ldots$ on the infinite path graph.

- **Open question:** What is the generating function for $\mathcal{F}(D)$? See On a square-ice analogue of plane partitions

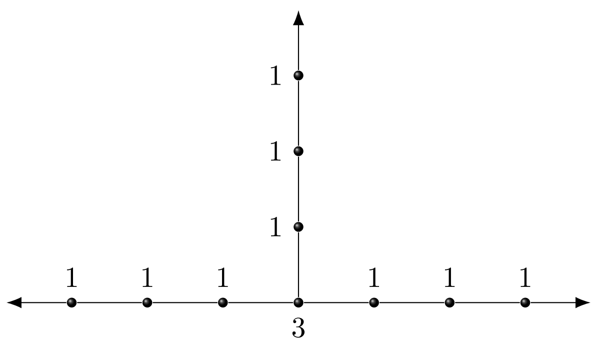
Open problem. We have seen that Young's lattice is the firing lattice for



The generating function for Young's lattice is $\prod_{i>0} \frac{1}{1-x^i}$. We have also seen that the lattice of shifted shapes (integer partitions with unequal parts) is the firing lattice for



Its generating function is $\prod_{i>0}(1+x^i)$. Notice that the divisor for Young's lattice comes from gluing together two copies of the divisor for the shifted shapes lattice. What if we glue together three copies:



Open problem. What is the generating function for this divisor?