Math 372 lecture for Wednesday, Week 13

Firing posets I.

Let $G = (V, E)$ be a connected, undirected multigraph with no loops. We assume that G is *locally finite*, meaning that there is a finite number of edges coming into and going out of each vertex. We also assume that not only is G connected, but that there is a finite path between any two vertices G. A *divisor* on G is a formal, possibly infinite, sum $D = \sum_{v\in V} D(v)v$ where each $D(v) \in \mathbb{Z}$. The *support* of D is $\operatorname{supp}(D) := \{v \in V : D(v) \neq 0\}$. If $\operatorname{supp}(D)$ is finite, then $\deg(D) := \sum_{v\in V} D(v)$. We think of D as an assignment of $D(v)$ dollars to each vertex v. Negative dollars are considered debt. Together, the set of divisors with vertex-wise addition forms an abelian group ($\mathbb{Z}$-module) $\operatorname{Div}(G)$. The *Laplacian* for G is the $\mathbb{Z}$-linear mapping $L\colon \operatorname{Div}(G) \to \operatorname{Div}(G)$ defined as follows for each vertex v (considered as a divisor): (i) if $w \neq v$, then $(Lv)(w) = -n_{vw}$ where n_{vw} is the number of directed edges from v to w, and (ii) $(Lv)(v) = \deg_G(v)$. Equivalently,

$$L(D)(v) := \sum_{vw\in E} (D(v) - D(w))v.$$

Local finiteness guarantees that L is well-defined. Fixing an ordering of the vertices, we can think of L as a (possibly infinite) matrix with rows and columns indexed by V.

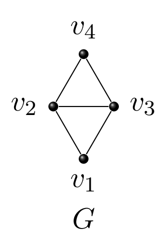
We can *fire* a vertex v on D to obtain a new divisor D' defined by

$$D' := D - Lv,$$

and we then write $D \xrightarrow{v} D'$. Firing v is called a *vertex-firing* or a *lending move* by v. We could also say that $D' = D + Lv$ is obtained from D by a *reverse-firing* or *lending move* at v, and write $D' \xrightarrow{-v} D$. Two divisors D and D' are *linearly equivalent*, written $D \sim D'$ if $D - D' \in \operatorname{im}(L)$. If $D \sim D'$, then there exists divisor σ such that $D' = D - L\sigma$. We call σ a firing script taking D to D' and write $D \xrightarrow{\sigma} D'$.

A vertex v is *unstable* in D if $D(v) \geq \deg_G(v)$. This means that after firing v, the vertex v will not be in debt. A vertex-firing is *legal* for D if v is unstable in D. We say D is *stable* if it has no unstable vertices, and D is stabilizable if there exists a finite sequence $v_1, v_2, \ldots$ of legal vertex firings leading to a stable divisor. (Question: what if we allowed infinite sequences?)

Definition 1. The *firing graph* for $D \in \operatorname{Div}(G)$ is the directed graph $\mathcal{F}(D)$ whose vertices are the divisors reachable from D by a finite sequence of legal vertex-firings, and with an edge from vertex H to vertex H' if there is a legal vertex firing taking H to H'.

Example. Consider the diamond graph  .

The firing graph for $D = 2v_1 + 2v_4 = (2, 0, 0, 2)$ is appears in Figure 1, and the firing graph for $D' = 2v_1 + v_2 + 2v_4 = (2, 1, 0, 2)$ appears in Figure 2.

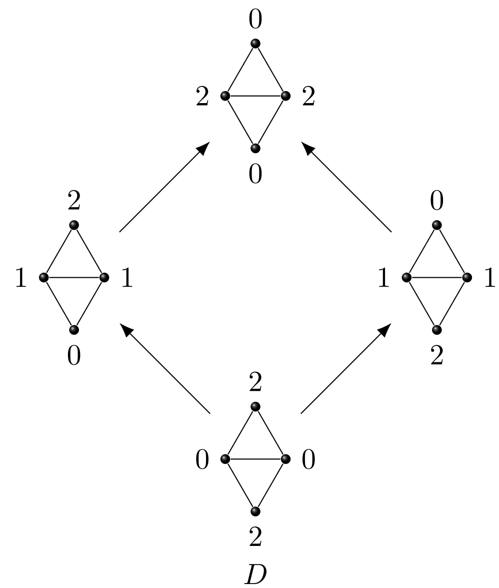


Figure 1: Firing graph for $D = (2, 0, 0, 2)$

Proposition 2. If G is finite, then the kernel of L is generated by $1_V = \sum_{v \in V} v$ (or the all-ones vector, thinking of L as a matrix).

Proof. Suppose $L(D) = 0$, and let

$$m = \max\{D(w) : w \in V\}.$$

Choose v such that $D(v) = m$. Since $L(D) = 0$, we have

$$0 = L(D)(v) = \sum_{vw \in E} (D(v) - D(w)) = \deg_G(v)m - \sum_{vw \in E} D(w).$$

Therefore,

$$m = \frac{1}{\deg_G(v)} \sum_{vw \in E} D(w).$$

However, since $D(w) \leq m$ for all $w \in V$, the above equality can only hold if $D(w) = m$ for all w adjacent to v. Then, since G is connected, it must be that $D(w) = m$ for all $w \in V$. $\square$

Corollary. Suppose G is finite, and let $D \in \operatorname{Div}(G)$.

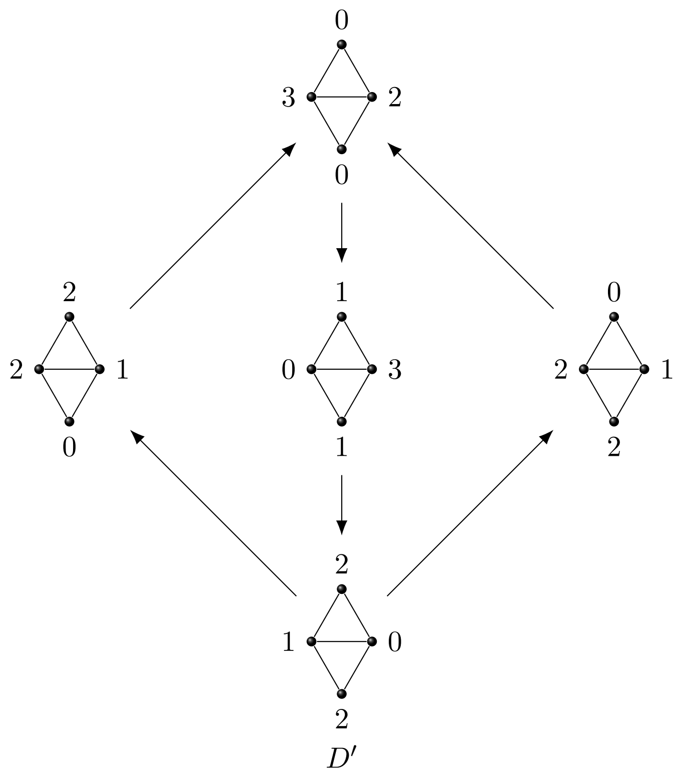


Figure 2: Firing graph for $D = (2, 1, 0, 2)$

1. Let C be a directed cycle in $\mathcal{F}(D)$ starting and ending at some divisor H. Say

$$H = H_1 \xrightarrow{v_1} H_2 \xrightarrow{v_2} \cdots \xrightarrow{v_k} H_{k+1} = H$$

   is a sequence of legal vertex firings corresponding to C. Then $\sum_{i=1}^{k} v_i = a1_V$ for some integer a.

2. Suppose $v_1, v_1, \dots, v_k$ and $w_1, \dots, w_\ell$ are both legal firing sequences taking D to the some divisor D'. Also suppose that neither firing sequence contains all of the vertices. Then the sequences are the same up to a permutation.

3. The firing poset is graded. The rank of $D' \in \mathcal{F}(D)$ is the number of vertices that must be fired from D to reach D' in the firing graph (which also equals the length of a smallest legal sequence of firings taking D to D').

Proof. For part 1, let $\sigma = \sum_{i=1}^{k} v_i$. Then $H = H - L\sigma$. It follows that $\sigma \in \ker(L)$, and hence σ is a multiple of 1_V. For part 2, let $\sigma := \sum_{i=1}^{k} v_i$ and $\tau = \sum_{i=1}^{\ell} w_i$. Then

$$D' = D - L\sigma = D - L\tau$$

implies that $\sigma - \tau \in \ker(L)$. Therefore, $\sigma = \tau + a 1_V$ for some integer a. Since $\tau(w) = 0$ for some $w \in V$, and $\sigma \geq 0$, it follows that $a \geq 0$. The since $\sigma(u) = 0$ for some $u \in V$ and $\tau \geq 0$, it follows that $a \leq 0$. Therefore, $a = 0$ and $\sigma = \tau$. Then since both σ and τ are nonnegative, the result follows.

Part 3 follows immediately from part 2. □

Theorem. (Least action principle) Let G be finite, and let $D \in \mathrm{Div}(G)$. Suppose that $v_1, \ldots, v_k$ is a sequence of legal vertex firings for D, and let $\sigma = \sum_{i=1}^k v_i$. Then if $D - L\tau$ is stable with $\tau \geq 0$, it follows that $\tau \geq \sigma$.

Proof. The proof goes by induction on k. The case $k = 0$, in which $\sigma = 0$, it obvious. So suppose $k > 0$. Since v_1 is unstable in D, in order for D to stabilize, v_1 must fire. Therefore, $v_1 \in \mathrm{supp}(\tau)$. Say $D \xrightarrow{v_1} D'$, and let $\tau' = \tau - v_1$. Then τ' stabilizes D' and $v_2, \ldots, v_k$ is a sequence of legal vertex firings. By induction $\sigma - v_1 \leq \tau'$, and the result follows:

$$\sigma \leq \tau' + v_1 = \tau.$$

□

Corollary. Let G be finite and $D \in \mathrm{Div}(G)$. Suppose that $v_1, \ldots, v_k$ and $w_1, \ldots, w_\ell$ are both legal firing sequences, and let $\sigma := \sum_{i=}^k v_i$ and $\tau := \sum_{i=1}^\ell w_i$ be the corresponding firing scripts. Further suppose that $D \xrightarrow{\sigma} D'$ and $D \xrightarrow{\tau} D''$ with both D' and D'' stable. Then $\sigma = \tau$ and $D' = D''$.

Proof. From the least action principle we have $\sigma \leq \tau$ and $\tau \leq \sigma$. □