Math 372 lecture for Wednesday, Week 10

The discrete Laplacian

Let $G = (V, E))$ be an undirected, connected, multigraph with vertex set $V = \{v_1, \dots, v_n\}$. For convenience, assume G has no loops. Last time, we defined the boundary mapping

$$\partial\colon \mathbb{Z}E \to \mathbb{Z}V,$$

determined by $\partial(e) = e^+ - e^-$ for each $e \in E$. Ordering the vertices and then using lexicographic ordering on the edges, as usual, boundary mapping becomes a matrix, and we can take its transpose

$$\partial^t\colon \mathbb{Z}V \to \mathbb{Z}E.$$

(More formally, define $\mathbb{Z}E^* := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}E^*, \mathbb{Z})$ to be the set of all $\mathbb{Z}$-linear mappings $\mathbb{Z}E \to \mathbb{Z}$, and identify $\mathbb{Z}E$ with $\mathbb{Z}E^*$ by $e \mapsto \chi_e$ for each $e \in E$ where χ_e is the characteristic function for e. Similarly, define $\mathbb{Z}V^*$ and identify it with $\mathbb{Z}V$. Then we get a mapping

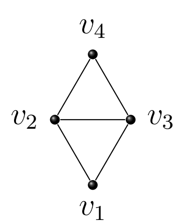
$$\mathbb{Z}V^* \xrightarrow{\partial^*} \mathbb{Z}E^*$$

defined by $\phi \mapsto \phi \circ \partial$ for each function $\phi\colon \mathbb{Z}V \to \mathbb{Z}$. Fixing our usual bases for $\mathbb{Z}V$ and $\mathbb{Z}E$ turns ∂ into a matrix and ∂^* into the transpose ∂^t.)

Definition. The *Laplacian* of G is the mapping (or matrix)

$$L := L(G) := \partial \circ \partial^t\colon \mathbb{Z}V \to \mathbb{Z}V.$$

Example. Consider the *diamond graph* G:



The Laplacian of G is

$$L = \partial\partial^t = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}.$$

The *(weighted) adjacency matrix* for G is the $n \times n$ matrix A where A_{ij} is the number of edges joining v_i to v_j. For each vertex $v \in V$, define $\deg_G(v)$ to be the number of edges with an endpoint equal to v. Then define the diagonal matrix $D = \operatorname{diag}(\deg_G(v_1), \cdots, \deg_G(v_n))$.

Proposition 1. The Laplacian of G is $L = D - A$.

Proof. Let $m = |E|$. For $1 \leq i < j \leq n$, we have

$$\begin{aligned}(\partial\partial^t)_{ij} &= \sum_{k=0}^{m} \partial_{ik}\partial^t_{kj} \\ &= \sum_{k=0}^{m} \partial_{ik}\partial_{jk} \\ &= -A_{ij} = L_{ij}\end{aligned}$$

since

$$\partial_{ik}\partial_{jk} = \begin{cases} -1 & \text{if } \{v_i, v_j\} \text{ is the } k\text{-th edge,} \\ 0 & \text{otherwise.}\end{cases}$$

Since L, D, and A are symmetric, we also have $L_{ij} = (\partial\partial^t)_{ij}$ for $i > j$. That leaves the case where $i = j$:

$$\begin{aligned}(\partial\partial^t)_{ii} &= \sum_{k=0}^{m} \partial_{ik}\partial^t_{ki} \\ &= \sum_{k=0}^{m} \partial_{ik}\partial_{ik} \\ &= D_{ii}\end{aligned}$$

since

$$\partial_{ik}\partial_{ik} = \begin{cases} 1 & \text{if } v_i \text{ is on the } k\text{-th edge,} \\ 0 & \text{otherwise.}\end{cases}$$

□

As an immediate corollary, we get:

Corollary 2. The sum of the rows and the sum of the columns of L are both $0 \in \mathbb{R}^n$.

(Another way to see the corollary: $L = \partial\partial^t$ implies that $\partial_0 \circ L = 0$ where ∂_0 is the 0-th boundary map, represented by the matrix $[1\ 1\ \ldots\ 1]$. (We will call ∂_0 the *degree* mapping, below.) This says the sum of the rows of L is 0. Since L is symmetric, the sum of the columns must then be 0, also.)

Definition. The *Picard group* of G is the abelian group

$$\mathrm{Pic}(G) := \mathbb{Z}V/\operatorname{im}(L).$$

Recall the exact sequence from last time:

$$0 \to \mathcal{C} \to \mathbb{Z}E \xrightarrow{\partial} \mathbb{Z}V \xrightarrow{\deg} \mathbb{Z} \to 0.$$

Here, if $f = \sum_{v\in V} a_v v$ for some $a_v \in \mathbb{Z}$, then $\deg(f) := \sum_{v\in V} a_v$. In particular, $\deg(v) = 1$, which would usually not be equal to $\deg_G(v)$, the number of edges incident on v, despite the similarity in notation. Let $\mathbb{Z}V_0$ denote the kernel of the degree mapping, i.e., the set of those $f = \sum_{v\in V} a_v v$ such that $\sum_{v\in V} a_v = 0$. The exact sequence says that $\operatorname{im}\partial \subseteq \mathbb{Z}V_0$. Then, since $L = \partial \circ \partial^t$, we have

$$\operatorname{im} L \subseteq \operatorname{im}\partial \subseteq \mathbb{Z}V_0.$$

Definition. The *Jacobian group* or *critical group* of G is

$$\mathrm{Jac}(G) := \mathbb{Z}V_0/\operatorname{im} L.$$

Proposition 3. Fix any $q \in V$. Then there is an isomorphism

$$\begin{aligned} \phi\colon\ \mathrm{Pic}(G) &\to \mathbb{Z} \oplus \mathrm{Jac}(G) \\ f &\mapsto (\deg(f), f - \deg(f)q). \end{aligned}$$

Proof. First notice that ϕ is well-defined: if $f, f' \in \mathbb{Z}V$ and $f = f' \bmod \operatorname{im} L$, write $f' = f + h$ where $h \in \operatorname{im} L$. We saw above that everything in the image of L has degree 0. Therefore,

$$\deg(f') = \deg(f+h) = \deg(f) + \deg(h) = \deg(f).$$

The inverse to ϕ is the mapping defined by $(d, g) \mapsto g + dq$. $\square$

Definition. Fix $q \in V$. The *reduced Laplacian* of G (with respect to q) is the matrix $\tilde{L}$ obtained by removing the row and column indexed by q from the Laplacian L.

Proposition 4. Fix $q = v_j \in V = \{v_1, \ldots, v_n\}$. Then there is a $\mathbb{Z}$-linear isomophism

$$\begin{aligned} \psi\colon\ \mathrm{Jac}(G) &\to \mathbb{Z}^{n-1}/\operatorname{im}(\tilde{L}) \\ \textstyle\sum_{i=1}^n a_i v_i \bmod \operatorname{im}(L) &\mapsto (a_1, \ldots, \hat{a_j}, \ldots, a_n) \bmod \operatorname{im}(\tilde{L}). \end{aligned}$$

Proof. Without loss of generality, take $j = 1$. We will first verify that ψ is well-defined. Think of L as an $n \times n$ matrix. For each i let the i-th column of L be denoted by ℓ_i and let $\tilde{\ell}_i$ be the vector ℓ_i with its first entry removed. Thus, the columns of the reduced Laplacian are $\tilde{\ell}_2, \ldots, \tilde{\ell}_n$. We need to show that $\psi(\ell_i) \in \operatorname{im}(\tilde{L})$ for each i. That follows immediately for $i > 1$, since $\psi(\ell_i) = \tilde{\ell}_i$. For $i = 1$, use Corollary 2, which says that $\sum_{i=1}^n \ell_i = 0$. It follows that $\ell_1 = -\sum_{i=2}^n \ell_i$ and thus

$$\psi(\ell_i) = \tilde{\ell}_1 = -\sum_{i=2}^n \tilde{\ell}_i \in \operatorname{im}(\tilde{L}).$$

maps to 0 under ψ if $i = j$ and maps to a column of $\tilde{L}$, otherwise. Therefore, ψ is well-defined.

To show ψ is an isomorphism, we exhibit its inverse:

$$\begin{aligned} \rho\colon \mathbb{Z}^{n-1}/\operatorname{im}(\tilde{L})\colon &\to \mathrm{Jac}(G) \\ (a_2, \ldots, a_n) \bmod \operatorname{im}(\tilde{L}) &\mapsto (-\textstyle\sum_{i=2}^n a_i, a_2, \ldots, a_n) \bmod \operatorname{im}(L). \end{aligned}$$

Reasoning as above, we see $\rho(\tilde{\ell}_i) = \ell_i$ for $i = 2, \ldots, n$. Thus, ρ is well-defined. Recalling that the sum of the coefficients of any element of $\mathrm{Jac}(G)$ is 0, it is clear that ρ is inverse to ψ. $\square$

Example. Let G be the diamond graph from the first example. Then

$$\mathrm{Pic}(G) \simeq \mathbb{Z}^4/\operatorname{Span}_{\mathbb{Z}}\{(2,-1,-1,0), (-1,3,-1,-1), (-1,-1,3,-1), (0,-1,-1,2)\}.$$

and

$$\mathrm{Jac}(G) \simeq \mathbb{Z}^3/\operatorname{Span}_{\mathbb{Z}}\{(3,-1,-1), (-1,3,-1), (-1,-1,2)\},$$

choosing $q = v_1$ in Proposition 3.