Math 372 lecture for Monday, Week 10

Cycles and cuts in graphs

Let $G = (V, E)$ be an undirected multigraph, not necessarily connected, and possibly containing loop edges. By *multigraph*, we mean that E can be a *multiset*, i.e., there may be more than one edge between a pair of vertices.

Definition 1. A *cycle* in a multigraph G is a path $C = v_1, e_1, v_2, e_2, \dots, v_n$ such $v_1 = v_n$, the first $n-1$ vertices are distinct, and no edge is repeated. We consider two cycles to be the same if they differ by a cyclic shift—i.e., cycles do not have distinguished starting points. Note that a loop is a cycle of length 1. We treat multiple edges between the same two vertices as distinct, so that the banana graph B_2, consisting of two vertices connected by two edges, is a cycle. A multigraph with no cycles is called *acyclic*.

In order to develop the algebraic theory of cycles and cuts, we need to choose an orientation $\mathcal{O}$ for the undirected multigraph G and consider the directed multigraph $(G, \mathcal{O})$. However, we will see that the main results are independent of the choice of orientation. If the vertices of G are ordered as $v_1, \dots, v_n$, then we will generally choose the *standard orientation*, which assigns the directed edge $e = (v_i, v_j)$ to the undirected edge $\{v_i, v_j\}$ whenever $i < j$. In this case, we write $e^+ = v_j$ and $e^- = v_i$.

Let $\mathbb{Z}E$ be the free abelian group on the undirected edges of G. In the case where E is a multiset, copies of edges are treated as distinct in $\mathbb{Z}E$. For example, if G is the banana graph B_2, mentioned above, then $\mathbb{Z}E \simeq \mathbb{Z}^2$. If $g = \sum_{e \in E} a_e\, e$ is an element of $\mathbb{Z}E$, then the *support* of g, denoted $\mathrm{supp}(g)$, is the set of edges e for which the coefficient a_e is nonzero.

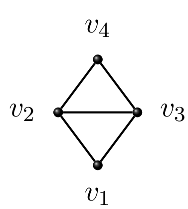
The orientation $\mathcal{O}$ allows us to define the *boundary* of an edge $e \in E$ as

$$\partial e := e^+ - e^- \in \mathbb{Z}V.$$

Extending linearly defines the *boundary map*,

$$\partial \colon \mathbb{Z}E \to \mathbb{Z}V.$$

If G is a simple graph, i.e., without multiple edges, then G is a 1-dimensional simplicial complex, and this ∂ is the usual simplicial boundary map. Fixing an ordering of the vertices and of the edges realizes ∂ as a matrix whose columns are indexed by the edges. See Figure 1 for an example.



$$\partial \;=\; \begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{array} \begin{array}{c} \begin{array}{ccccc} \overline{\mathbf{12}} & \overline{\mathbf{13}} & \overline{\mathbf{23}} & \overline{\mathbf{24}} & \overline{\mathbf{34}} \end{array} \\ \begin{pmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Figure 1. The oriented incidence matrix with respect to the standard orientation. Rows and columns are labeled with vertex indices.

The cycle space. Consider a cycle $C = u_0, e_1, u_1, e_2, \dots, e_k, u_k$ in the undirected graph G. The *sign* of an edge $e \in E$ with respect to C and the orientation $\mathcal{O}$ is $\sigma(e, C) = 1$ if $C = v, e, v$ is a loop at a vertex v, and otherwise

$$\sigma(e,C) = \begin{cases} 1 & \text{if } e^- = u_i \text{ and } e^+ = u_{i+1} \text{ for some } i, \\ -1 & \text{if } e^+ = u_i \text{ and } e^- = u_{i+1} \text{ for some } i, \\ 0 & \text{otherwise } (e \text{ does not occur in } C). \end{cases}$$

We then identify C with the formal sum $\sum_{e \in E} \sigma(e, C)e \in \mathbb{Z}E$. For notational convenience, if $e = uv$, we denote $-e$ by vu.

Example 2. For the graph in Figure 1 (with the standard orientation), the cycle

$$C = v_1, \{v_1, v_2\}, v_2, \{v_2, v_3\}, v_3, \{v_1, v_3\}, v_1$$

is identified with

$$C = \overline{12} + \overline{23} - \overline{13} = \overline{12} + \overline{23} + \overline{31} \in \mathbb{Z}E$$

where $\overline{12} := v_1 v_2$, etc. This cycle is shown on the left in Figure 3.

Definition 3. The *(integral) cycle space*, $\mathcal{C} \subset \mathbb{Z}E$, is the $\mathbb{Z}$-span of all cycles.

Example 4. Let G be the oriented graph pictured in Figure 2. The cycle space is isomorphic to $\mathbb{Z}^2$ with basis $e_1 - e_2$, e_3.

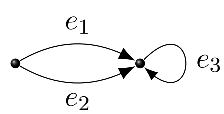


Figure 2. $\mathcal{C} \simeq \mathbb{Z}^2$.

The cut space. A *directed cut* of G is an ordered partition of the vertices into two nonempty parts. For each nonempty $U \subsetneq V$, we get the directed cut, (U, U^c). The *cut-set* corresponding to U, denoted c_U^*, is the collection of edges with one vertex in U and the other in the complement, U^c. For each $e \in E$, define the *sign* of e in c_U^* with respect to the orientation $\mathcal{O}$ by

$$\sigma(e, c_U^*) = \begin{cases} 1 & \text{if } e^- \in U \text{ and } e^+ \in U^c, \\ -1 & \text{if } e^+ \in U \text{ and } e^- \in U^c, \\ 0 & \text{otherwise } (e \text{ does not occur in } c_U^*). \end{cases}$$

We identify the cut-set c_U^* with the formal sum $\sum_{e \in E} \sigma(e, c_U^*)e \in \mathbb{Z}E$. Thus, for instance, $c_{U^c}^* = -c_U^*$. If G is not connected, there will be empty cut-sets, identified with $0 \in \mathbb{Z}E$. A *vertex cut* is the cut-set corresponding to a single vertex, $U = \{v\}$, and we write c_v^* for c_U^* in that case. A minimal nonempty cut-set with respect to inclusion is called a *bond*. For example, the cut-set $c_{\{v_2, v_3\}}^*$ in Example 5 is not a bond.

Example 5. For the graph in Figure 1 (with the standard orientation), the cut-set corresponding to $\{v_2, v_3\}$ is

$$\begin{aligned} c^*_{\{v_2,v_3\}} &= -\overline{12} - \overline{13} + \overline{24} + \overline{34} \\ &= \ \overline{21} + \overline{31} + \overline{24} + \overline{34} \in \mathbb{Z}E. \end{aligned}$$

It is shown on the right in Figure 3.

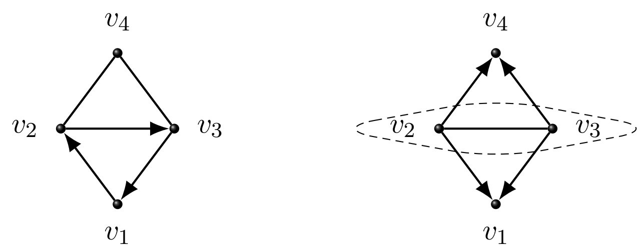


Figure 3. The cycle C from Example 2 (left) and the cut-set $c^*_{\{v_2,v_3\}}$ from Example 5 (right).

Definition 6. The *(integral) cut space*, $\mathcal{C}^* \subset \mathbb{Z}E$, is the $\mathbb{Z}$-span of all cut-sets.

Exercise 7. If U is a nonempty subset of $V(G)$, the *subgraph of G induced by U*, denoted $G[U]$, is the graph with vertex set U and edge multiset consisting of those edges with both ends in U. If G is connected, show that the cut-set corresponding to a nonempty set $U \subsetneq V(G)$ is a bond if and only if $G[U]$ and $G[U^c]$ are connected. If G is not connected, show that its bonds are exactly the bonds of its connected components.

Bases for cycle and cut spaces. A *spanning forest* for G is a maximal subset $F \subseteq E$ that contains no cycles, and such that every vertex of G is on some edge in F. A *spanning tree* for G is a connected spanning forest. (Thus, a spanning forest consists of the union of spanning trees, one for each connected component of the graph.)

Proposition 8. *Suppose that G is a multigraph on n vertices. The following are equivalent:*

(1) G is a tree;

(2) G is minimal connected: G is connected and removing any edge from G yields a disconnected multigraph;

(3) G is maximal acyclic: G is acyclic and adding any edge between vertices of G produces a cycle;

(4) G is connected and has $n-1$ edges;

(5) G is acyclic and has $n-1$ edges.

Fix a spanning forest F for G, and for notational purposes, identify F with its set of edges. Let $F^c := E \setminus F$.

Exercise 9. Show that for each $e \in F^c$, the graph with edges $F \cup \{e\}$ has a unique cycle, c_e, such that $\sigma(e, c_e) = 1$. (This holds even if e is a loop.)

Pick $e \in F$. The forest F is a disjoint union of spanning trees of the connected components of G, and one of these spanning trees, say T, contains e. Removing e disconnects T into two connected components T^- and T^+ where e^- is contained in T^-. Let U be the vertices of T^-. Define the cut-set $c_e^* := c_U^*$, and note that $\sigma(e, c_e^*) = 1$.

Exercise 10. Show that the cut-sets c_e^* are bonds.

Theorem 11.

(1) The kernel of the boundary mapping is the cycle space: $\ker \partial = \mathcal{C}$.

(2) Let F be a spanning forest of G. Then $\{c_e : e \in F^c\}$ is a $\mathbb{Z}$-basis for $\mathcal{C}$ and $\{c_e^* : e \in F\}$ is a $\mathbb{Z}$-basis for $\mathcal{C}^*$.

(3) $\operatorname{rank}_{\mathbb{Z}} \mathcal{C} = |E| - |V| + \kappa$, and $\operatorname{rank}_{\mathbb{Z}} \mathcal{C}^* = |V| - \kappa$ where κ is the number of connected components of G.

(4) $\mathcal{C} = (\mathcal{C}^*)^\perp := \{f \in \mathbb{Z}E : \langle f, g \rangle = 0$ for all $g \in \mathcal{C}^*\}$ where $\langle\ ,\ \rangle$ is defined for $e, e' \in E$ by

$$\langle e, e' \rangle := \delta(e, e') = \begin{cases} 1 & \text{if } e = e', \\ 0 & \text{if } e \neq e' \end{cases}$$

and extended linearly for arbitrary pairs in $\mathbb{Z}E$.

(5) If G is connected, then the following sequence is exact, i.e., the image of each mapping is equal to the kernel of the mapping that follows it:

$$0 \longrightarrow \mathcal{C} \longrightarrow \mathbb{Z}E \xrightarrow{\partial} \mathbb{Z}V \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0.$$

Proof. It is clear that $\mathcal{C} \subseteq \ker \partial$. For the opposite inclusion, consider an arbitrary $f = \sum_{e \in E} a_e e \in \mathbb{Z}E$. Fix a spanning forest F, and define

$$g := f - \sum_{e \in F^c} a_e c_e.$$

Then $\partial f = \partial g$ and $\operatorname{supp}(g) \subseteq F$. If $g \neq 0$, then the union of the edges in $\operatorname{supp}(g)$ is a subforest of F with at least one edge; choose a leaf vertex v in this subforest. Then $v \in \operatorname{supp}(\partial g)$, and hence, $\partial f = \partial g \neq 0$. So if $f \in \ker \partial$, then $g = 0$, i.e.,

$$f = \sum_{e \in F^c} a_e c_e \in \mathcal{C}.$$

The proves part 1.

For part 2, we have just seen that $\{c_e : e \in F^c\}$ spans $\ker \partial$. These elements are linearly independent since $c_e \cap F^c = \{e\}$. To see that $\{c_e^* : e \in F\}$ is a basis for $\mathcal{C}^*$, first note that each cut-set is a linear combination of vertex cuts:

Exercise 12. Show that for each nonempty $U \subsetneq V$,

$$c_U^* = \sum_{v \in U} c_v^*.$$

Hence, the vertex cuts span $\mathcal{C}^*$. However, each vertex cut is a linear combination of the c_e^*:

Exercise 13. Show that for each $v \in V$,

$$c_v^* = \sum_{e \in F : e^- = v} c_e^* - \sum_{e \in F : e^+ = v} c_e^*.$$

(One way to proceed: First argue that we may assume G is connected. Then analyze the above expression in terms of the components of F after removing all edges incident on v.)

Hence, $\{c_e^* : e \in F\}$ spans the cut space. Linear independence of the c_e^* follows from the fact that $c_e^* \cap F = \{e\}$.

For part 3, first note that we have just shown that both the cycle and cut space have $\mathbb{Z}$-bases, hence it makes sense to consider their ranks. Since the number of edges in a tree is one less than the number of its vertices, we have

$$\operatorname{rank}_{\mathbb{Z}} \mathcal{C}^* = |F| = |V| - \kappa,$$

and

$$\operatorname{rank}_{\mathbb{Z}} \mathcal{C} = |F^c| = |E| - |F| = |E| - |V| + \kappa.$$

For part 4, let $f = \sum_{e \in E} a_e e$. For each $v \in V$,

$$\langle f, c_v^* \rangle = \sum_{e : e^- = v} a_e - \sum_{e : e^+ = v} a_e,$$

which is the negative of the coefficient of v in ∂f. Thus, $\langle f, c_v^* \rangle = 0$ for all $v \in V$ if and only if $f \in \ker \partial = \mathcal{C}$. Since the vertex cuts span $\mathcal{C}^*$, the result follows.

Part 5 is left as an exercise. □

Remark 14. The number $\operatorname{rank}_{\mathbb{Z}} \mathcal{C}$ is known as the *cycle rank* or *cyclomatic number* of G.