Math 372 lecture for Friday, Week 10

Quick introduction to Smith normal form

These notes are intended as help with the homework due next Wednesday. The complete context for the material presented here will be presented over the next couple of days.

Let M be an $m \times n$ matrix with integer coefficients, and consider the *cokernel* of M, defined by

$$\operatorname{cok}(M) := \mathbb{Z}^m / \operatorname{im} M.$$

So the cokernel of M is the set of integer vectors $(a_1, \dots, a_m)$ for which we add vectors as usual, but such that any vector that is a column of M is thought of as the zero vector.

Example. Let $M = [5]$, a 1×1 matrix. Then $\operatorname{cok}(M) = \mathbb{Z}/5\mathbb{Z}$.

Example. Let $M = \operatorname{diag}(2, 3)$, a 2×2 diagonal matrix. Then

$$\begin{aligned} \operatorname{cok}(M) = \mathbb{Z}^2 / \operatorname{Span}\left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right\} &\xrightarrow{\sim} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \\ (a, b) &\mapsto (a \bmod 2, b \bmod 3). \end{aligned}$$

Setting $(2, 0)$ equal to $(0, 0)$ in $\mathbb{Z}^2$, just means we can work modulo 2 in the first coordinate. Similarly, we can work modulo 3 in the second coordinate.

Example. Let $M = \operatorname{diag}(0, 0, 1, 2, 3)$. Then

$$\begin{aligned} \operatorname{cok}(M) \simeq \mathbb{Z}/0\mathbb{Z} \oplus \mathbb{Z}0\mathbb{Z} \oplus \mathbb{Z}/1\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} &\xrightarrow{\sim} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \\ (a, b, c, d, e) &\mapsto (a, b, d, e). \end{aligned}$$

Here, we use the fact that $\mathbb{Z}/0\mathbb{Z} = \mathbb{Z}$ and $\mathbb{Z}/1\mathbb{Z} = \{0\}$. For instance, in $\operatorname{cok}(M)$, the third coordinate is always equivalent to 0 modulo 1, hence, we can drop that coordinate in our isomorphism.

Definition. The *integer row (resp., column) operations* on an integer matrix consist of the following:

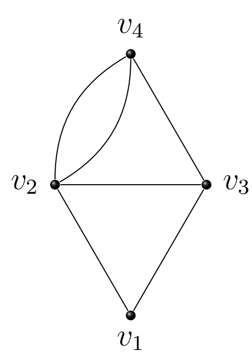
1. swapping two rows (resp., columns);
2. negating a row (resp., column);
3. adding one row (resp., column) to a different row (resp., column).

Claim. By performing integer row and column operations, the matrix M can be transformed into a diagonal matrix D, i.e., $D_{ij} = 0$ for $i \neq j$. To make the final form unique, one may insist that the diagonal elements satisfy $D_{i,i} | D_{i+1,i+1}$ for all i. Start with the the identity matrix I_m and perform all of the same row operations on I_m as used in the reduction of M to D to create a matrix U. Similarly, start with I_n and perform the same column operations on it as used in the reduction of M to D to create a matrix V. Then both U and V have inverses that are integer matrices (equivalently, $\det(U) = \pm 1$ and $\det(V) = \pm 1$), and

$$UMV = D.$$

Roughly, the algorithm for reduction to a diagonal matrix goes like this: Use column operations to put the gcd of the elements in the first row into the $1,1$-position of the matrix. Then use the first column to make the other entries in the first row equal to 0. Continuing, use row operations to put the gcd of the first column into the $1,1$-position, Then use row operations to make the other entries in the first column equal to 0. By this time, you may have put nonzero entries in the first row again. Repeat. Eventually, every entry in the first row and column besides the $1,1$-entry will be 0. Proceed inductively: now only use row and column operations not involving the first row and column.

Example. Let's figure out the structure of $\mathrm{Pic}(G)$ for the graph pictured below:



The Laplacian matrix is

$$L = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 4 & -1 & -2 \\ -1 & -1 & 3 & -1 \\ 0 & -2 & -1 & 3 \end{pmatrix}.$$

Perform integer row and column operations to diagonalize L:

$$\begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 4 & -1 & -2 \\ -1 & -1 & 3 & -1 \\ 0 & -2 & -1 & 3 \end{pmatrix} \xrightarrow{c_1 \to c_1 + c_2} \begin{pmatrix} 1 & -1 & -1 & 0 \\ 3 & 4 & -1 & -2 \\ -2 & -1 & 3 & -1 \\ -2 & -2 & -1 & 3 \end{pmatrix}$$

$$\xrightarrow[c_3 \to c_3 + c_1]{c_2 \to c_2 + c_1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 7 & 2 & -2 \\ -2 & -3 & 1 & -1 \\ -2 & -4 & -3 & 3 \end{pmatrix}$$

$$\xrightarrow[r_3 \to r_3 + 2r_1, r_4 \to r_4 + 2r_1]{r_2 \to r_2 - 3r_1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 7 & 2 & -2 \\ 0 & -3 & 1 & -1 \\ 0 & -4 & -3 & 3 \end{pmatrix}$$

$$\xrightarrow{c_2 \to c_2 - 3c_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -2 \\ 0 & -6 & 1 & -1 \\ 0 & 5 & -3 & 3 \end{pmatrix}$$

$$\xrightarrow[c_4 \to c_4 + 2c_2]{c_3 \to c_3 - 2c_2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -6 & 13 & -13 \\ 0 & 5 & -13 & 13 \end{pmatrix}$$

$$\xrightarrow[r_4 \to r_4 - 5r_2]{r_3 \to r_3 + 6r_2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 13 & -13 \\ 0 & 0 & -13 & 13 \end{pmatrix}$$

$$\xrightarrow{c_4 \to c_4 + c_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 13 & 0 \\ 0 & 0 & -13 & 0 \end{pmatrix}$$

$$\xrightarrow{r_4 \to r_4 + r_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 13 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Apply the row operations above to I_4 to get U and apply the column operations to I_4 to get V:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_3 \to r_3 + 2r_1, r_4 \to r_4 + 2r_1]{r_2 \to r_2 - 3r_1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[r_4 \to r_4 - 5r_2]{r_3 \to r_3 + 6r_2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -16 & 6 & 1 & 0 \\ 17 & -5 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{r_4 \to r_4 + r_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -16 & 6 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} = U.$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{c_1 \to c_1 + c_2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[c_3 \to c_3 + c_1]{c_2 \to c_2 + c_1} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{c_2 \to c_2 - 3c_3} \begin{pmatrix} 1 & -2 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[c_4 \to c_4 + 2c_2]{c_3 \to c_3 - 2c_2} \begin{pmatrix} 1 & -2 & 5 & -4 \\ 1 & -1 & 3 & -2 \\ 0 & -3 & 7 & -6 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{c_4 \to c_4 + c_3} \begin{pmatrix} 1 & -2 & 5 & 1 \\ 1 & -1 & 3 & 1 \\ 0 & -3 & 7 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = V.$$

We then have

$$\begin{aligned} ULV &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -16 & 6 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 4 & -1 & -2 \\ -1 & -1 & 3 & -1 \\ 0 & -2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 & 5 & 1 \\ 1 & -1 & 3 & 1 \\ 0 & -3 & 7 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 13 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Therefore,

$$\mathrm{Pic}(G) = \mathrm{cok}(L) \simeq \mathbb{Z}/1\mathbb{Z} \oplus \mathbb{Z}/1\mathbb{Z} \oplus \mathbb{Z}/13\mathbb{Z} \oplus \mathbb{Z} \simeq \mathbb{Z} \oplus \mathbb{Z}/13\mathbb{Z}.$$

The explicit isomorphism would be essentially given by the matrix U (as we will see) and goes like this

$$\begin{array}{ccc} \mathrm{Pic}(G) = \mathbb{Z}^4/\operatorname{im}(L) \to \mathbb{Z}/1\mathbb{Z} \oplus \mathbb{Z}/1\mathbb{Z} \oplus \mathbb{Z}/13\mathbb{Z} \oplus \mathbb{Z} & \to & \mathbb{Z}/13\mathbb{Z} \oplus \mathbb{Z} \end{array}$$

$$(a,b,c,d) \mapsto U \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ -3a+b \\ -16a+6b+c \\ a+b+c+d \end{pmatrix} \mapsto (-16a+6b+c, a+b+c+d).$$

You can check that each column of L is sent to $(0,0)$ under this mapping, and thus the mapping is well-defined.

To find the structure of $\mathrm{Jac}(G)$, first take the reduced Laplacian with respect to any vertex, then apply the procedure illustrated above. For instance, the reduced Laplacian with respect to v_1 is

$$\tilde{L} = \begin{pmatrix} 4 & -1 & -2 \\ -1 & 3 & -1 \\ -2 & -1 & 3 \end{pmatrix}.$$

The diagonalized version of $\tilde{L}$ will be

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 13 \end{pmatrix},$$

and so $\mathrm{Jac}(G) \simeq \mathbb{Z}/13\mathbb{Z}$.