Math 372 lecture for Wednesday, Week 9

Simplicial complexes: examples

Let $\Delta \subset 2^{[n]}$ be a d-dimensional simplicial complex. For each $i \in \mathbb{Z}$ we have the space of i-dimensional chains $C_i := C_i(\Delta, \mathbb{Q}) := \mathbb{Q}F_i$, the vector space of formal sums of i-dimensional faces of Δ. We have $C_i = 0$ for $i < -1$ and $i > d$, and since $F_{-1} = \{\emptyset\}$, we have $C_{-1} = \mathbb{Q}$. The boundary mapping $\partial_i \colon C_i \to C_{i-1}$ is defined as follows: if $\sigma = \overline{\sigma_1 \dots \sigma_{i+1}} \in F_i$ where the σ_k are the vertices of σ and $\sigma_1 < \cdots < \sigma_{i+1}$, then

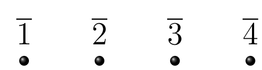
$$\begin{aligned}\partial_i(\sigma) &= \sum_{k=1}^{i+1} (-1)^{k-1} \overline{\sigma_1 \dots \widehat{\sigma_k} \dots \sigma_{i+1}} \\ &= \overline{\sigma_2\sigma_3\sigma_4 \dots \sigma_{i+1}} - \overline{\sigma_1\sigma_3\sigma_4 \dots \sigma_{i+1}} + \overline{\sigma_1\sigma_2\sigma_4 \dots \sigma_{i+1}} + \cdots .\end{aligned}$$

for $-1 \leq 1 \leq d$ and $\partial_i = 0$, otherwise. Recall the definitions of the reduced homology groups and Betti numbers:

$$\begin{aligned}\widetilde{H}_i &:= \widetilde{H}_i(\Delta) := \ker \partial_i / \operatorname{im} \partial_{i+1} \\ \tilde{\beta}_i &:= \tilde{\beta}_i(\Delta) := \dim \widetilde{H}_i = \operatorname{nullity}(\partial_i) - \operatorname{rank}(\partial_{i+1}).\end{aligned}$$

Examples.

I.1.



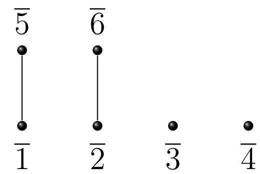
$$0 \to \mathbb{Q}^4 \xrightarrow[\partial_0]{\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}} \mathbb{Q} \to 0$$

We have $\operatorname{rank}(\partial_0) = 1$, and hence by the rank-nullity theorem, nullity $= 4 - 1 = 3$. The only non-vanishing homology group is

$$\begin{aligned}\widetilde{H}_0(\Delta) &= \ker \partial_0 / \operatorname{im} \partial_1 = \ker \partial_0 = \operatorname{Span}\left\{\overline{2} - \overline{1}, \overline{3} - \overline{1}, \overline{4} - \overline{1}\right\} \\ \tilde{\beta}_0 &= 3.\end{aligned}$$

Question: What are the homology groups and Betti numbers for $\Delta = \left\{\overline{1}, \dots, \overline{n}\right\}$ for general $n \geq 1$?

I.2.



$$0 \to \mathbb{Q}^2 \xrightarrow[\partial_1]{\begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}} \mathbb{Q}^6 \xrightarrow[\partial_0]{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}} \mathbb{Q} \to 0$$

We have

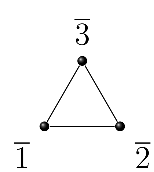
$$\begin{aligned} \operatorname{rank}(\partial_0) = 1, \operatorname{nullity}(\partial_0) = 6 - 1 = 5 \\ \operatorname{rank}(\partial_1) = 2, \operatorname{nullity}(\partial_1) = 0. \end{aligned}$$

Therefore, $\tilde{\beta}_0 = 5 - 2 = 3$ and $\tilde{\beta}_1 = 0$. The same as in example I.
Homology:

$$\begin{aligned} \widetilde{H}_0 = \ker \partial_0 / \operatorname{im} \partial_1 &= \operatorname{Span}\{\overline{2}-\overline{1}, \overline{3}-\overline{1}, \overline{4}-\overline{1}, \overline{5}-\overline{1}, \overline{6}-\overline{1}\} / \operatorname{Span}\left\{\overline{5}-\overline{1}, \overline{6}-\overline{2}\right\} \\ &\simeq \operatorname{Span}\{\overline{2}-\overline{1}, \overline{3}-\overline{1}, \overline{4}-\overline{1}\} \end{aligned}$$

In $\widetilde{H}_0$, we have $\overline{5} = \overline{1}$ and $\overline{6} = \overline{2}$, which means, $\overline{5} - \overline{1} = 0$ and $\overline{6} - \overline{1} = \overline{2} - \overline{1}$.
Question: How does this example generalize?

II.1



$$0 \to \mathbb{Q}^3 \xrightarrow[\partial_1]{\begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}} \mathbb{Q}^3 \xrightarrow[\partial_0]{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}} \mathbb{Q} \to 0$$

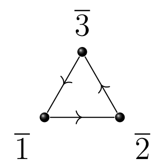
We have

$$\begin{aligned} \operatorname{rank}(\partial_0) &= 1, \operatorname{nullity}(\partial_0) = 3 - 1 = 2 \\ \operatorname{rank}(\partial_1) &= 2, \operatorname{nullity}(\partial_1) = 3 - 2 = 1. \end{aligned}$$

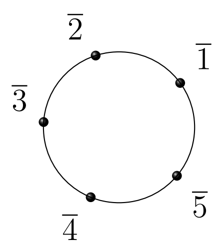
Therefore, $\tilde{\beta}_0 = 2 - 2 = 0$ and $\tilde{\beta}_1 = 1$.
Homology:

$$\widetilde{H}_1 = \ker \partial_1 / \operatorname{im} \partial_2 = \ker \partial_1 = \operatorname{Span}\{\overline{23} - \overline{13} + \overline{12}\}.$$

A picture of the generator for $\widetilde{H}_1$:



II.2.



$$0 \to \mathbb{Q}^5 \xrightarrow[\partial_1]{\begin{pmatrix} -1 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}} \mathbb{Q}^3 \xrightarrow[\partial_0]{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}} \mathbb{Q} \to 0$$

We have

$$\begin{aligned} \operatorname{rank}(\partial_0) &= 1, \operatorname{nullity}(\partial_0) = 5 - 1 = 4 \\ \operatorname{rank}(\partial_1) &= 4, \operatorname{nullity}(\partial_1) = 5 - 4 = 1. \end{aligned}$$

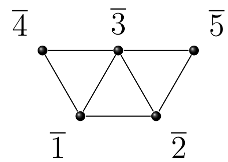
Therefore, $\tilde{\beta}_0 = 4 - 4 = 0$ and $\tilde{\beta}_1 = 1$.
Homology:

$$\widetilde{H}_1 = \ker \partial_1 / \operatorname{im} \partial_2 = \ker \partial_1 = \operatorname{Span}\{\overline{12} + \overline{23} + \overline{34} + \overline{45} - \overline{15}\}$$

The first homology is generated by a cycle of edges.

Question: What happens in homology if we start with the triangle and subdivide its edges arbitrarily?

III.1.



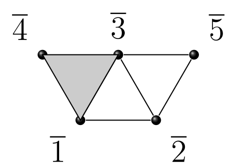
Let's compute the first homology.

$$0 \xrightarrow[\partial_2]{} \mathbb{Q}^7 \xrightarrow[\partial_1]{\begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}} \mathbb{Q}^5$$

We have rank $\partial_1 = 4$, and so nullity$\partial_1 = 7 - 4 = 3$. Therefore $\tilde{\beta}_1 = 3$. The homology is generated by the cycles surrounding the three bounded faces of the complex as drawn above:

$$\widetilde{H}_1 = \ker \partial_1 = \text{Span}\{\overline{23} - \overline{13} + \overline{12}, \overline{34} - \overline{14} + \overline{13}, \overline{35} - \overline{25} + \overline{23}\}$$

III.2.



For first homology, note that $\partial_2(\overline{134}) = \overline{34} - \overline{14} + \overline{13}$. We get

$$\mathbb{Q} \xrightarrow[\partial_2]{\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}} \mathbb{Q}^7 \xrightarrow[\partial 1]{\begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}} \mathbb{Q}^5$$

We have $\text{nullity}(\partial_1) = 7 - 4 = 3$ and $\operatorname{rank} \partial_2 = 1$. Therefore $\tilde{\beta}_1 = 3 - 1 = 2$. The homology is generated by the cycles surrounding the two unfilled bounded faces of the complex:

$$\widetilde{H}_1 = \ker \partial_1 \operatorname{im} \partial_2 = \text{Span}\{\overline{23} - \overline{13} + \overline{12}, \overline{34} - \overline{14} + \overline{13}, \overline{35} - \overline{25} + \overline{23}\} / \text{Span}\left\{\overline{34} - \overline{14} + \overline{13}\right\}$$

$$\simeq \text{Span}\{\overline{23} - \overline{13} + \overline{12}, \overline{35} - \overline{25} + \overline{23}\}$$

The cycle $\overline{34} - \overline{14} + \overline{13}$ is the boundary of the shaded face, and thus has become 0 in the homology group.

Question: How does this example generalize?

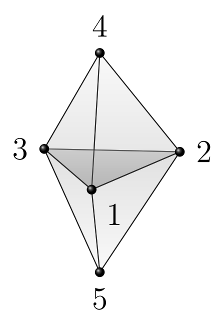
Exercise. Draw the following simplicial complexes, determine their Betti numbers, and describe bases for their homology groups. Recall that a *facet* of a simplicial complex is a face that is maximal with respect to inclusion. So we can describe a simplicial complex by just listing its facets. The whole simplicial complex then consists of the facets and all of their subsets.

1. Δ with facets $\overline{123}$, $\overline{24}$, $\overline{34}$, $\overline{45}$, $\overline{56}$, $\overline{57}$, $\overline{89}$, $\overline{10}$.

2. Δ with facets $\overline{123}$, $\overline{14}$, $\overline{24}$, and $\overline{34}$.

3. Δ with facets $\overline{123}$, $\overline{124}$, $\overline{134}$, $\overline{234}$, $\overline{125}$, $\overline{135}$, $\overline{235}$. (Two hollow tetrahedra glued along a face.)

Let Δ be the simplicial complex from Exercise 3, above. It is called the *equatorial bipyramid*:



In each dimension i except 2, every i-cycle is an i-boundary. For instance, in dimension 0, we have that $\overline{5} - \overline{4}$ is a 0-cycle since its boundary is $\overline{\emptyset} - \overline{\emptyset} = 0$. However, since $\overline{14}$ and $\overline{15}$ are part of the simplicial complex, we see

$$\partial_i(-\overline{15} + \overline{14}) = -(\overline{5} - \overline{1}) + (\overline{4} - \overline{1}) = \overline{5} - \overline{4}$$

is a boundary. Thus, in $\widetilde{H}_1$, we have that $\overline{5} - \overline{4} = 0$. Similarly,

$$\overline{14} - \overline{34} + \overline{35} - \overline{15}$$

is a 1-cycle of Δ, and it is the boundary of $-\overline{134} + \overline{135}$.

However, $\widetilde{H}_2 \simeq \mathbb{Q}^2$. Notice, that the solid tetrahedra $\overline{1234}$ and $\overline{1235}$ are not part of Δ. We can still imagine there boundaries, though, and these are generators for $\widetilde{H}_2$. Their boundaries will be 2-cycles, since $\partial^2 = 0$, however, these 2-cycles are not boundaries of Δ. So they are nonzero in $\widetilde{H}_2$.

If we added $\overline{1234}$ to Δ, we would get a simplicial complex with $\widetilde{H}_2 \simeq \mathbb{Q}$.

Let Δ be a simplicial complex, and consider the homology of Δ with coefficients in $\mathbb{Z}$. This means that $C_i := \mathbb{Z}F_i$ and the boundary mappings $\partial_i \colon \mathbb{Z}F_i \to \mathbb{Z}F_{i-1}$ are only $\mathbb{Z}$-linear.

For a hint at the difference between $\mathbb{Z}$-coefficients and $\mathbb{Q}$-coefficients, note that

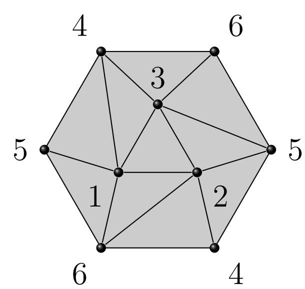
$$\mathbb{Q}^2/\operatorname{Span}_{\mathbb{Q}}\{(0,2)\} \simeq \mathbb{Q}, \quad \text{and} \quad \mathbb{Z}^2/\operatorname{Span}_{\mathbb{Z}}\{(0,2)\} \simeq \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

The space $\mathbb{Z}^2/\operatorname{Span}_{\mathbb{Z}}\{(0,2)\}$ is a $\mathbb{Z}$-module, meaning it is like a vector space but with the scalars being the ring $\mathbb{Z}$ rather than a field. It has a *free* part isomorphic to $\mathbb{Z}$ and a *torsion* part, isomorphic to $\mathbb{Z}/2\mathbb{Z}$. In general, if $v_1, \ldots, v_k \in \mathbb{Z}^n$,

$$M := \mathbb{Z}^n/\operatorname{Span}_{\mathbb{Z}}\{v_1, \ldots, v_k\} \simeq \mathbb{Z}^r \oplus (\mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_\ell\mathbb{Z})$$

for some r and some integers $d_1, \ldots, d_\ell$. The *free part* of M is $\mathbb{Z}^r$, and we say M has *rank* r. The *torsion part* of M is $(\mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_\ell\mathbb{Z})$.

To start to get an idea of the difference for simplicial complexes, let Δ be the triangulation of the real projective plane given below:

$\Delta =$ 

Where edges appear twice, it is understood that they should be glued together. This space might look contractible (although something complicated is clearly happening on the boundary), and in fact, over $\mathbb{Q}$, all of its homology groups are trivial. Working over the integers, though, something interesting happens in dimension 2. Consider orienting all ten triangles in the counterclockwise direction and adding them to get a 2-chain α. We have

$$\begin{aligned}\partial_2(\alpha) &= \overline{45} + \overline{56} - \overline{46} + \overline{45} + \overline{56} - \overline{46} \\ &= 2\left(\overline{45} + \overline{56} - \overline{46}\right).\end{aligned}$$

Letting $\beta := \overline{45} + \overline{56} - \overline{46}$, we have that $2\beta \in \operatorname{im}(\partial_2)$. Therefore, $2\beta = 0 \in \widetilde{H}_2 := \ker \partial_1 / \operatorname{im} \partial_2$. If we were working over $\mathbb{Q}$, this would also imply that β is 0 in $\widetilde{H}_2$ (since $\partial(\alpha/2) = \beta$). But over $\mathbb{Z}$, it turns out that β is nonzero in homology. In fact, we have

$$\widetilde{H}_2(\Delta, \mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}.$$