Simplicial complexes

1. First definitions.

An (abstract) *simplicial complex* Δ on a finite set S is a collection of subsets of S, closed under the operation of taking subsets. The elements of a simplicial complex Δ are called *faces*. An element $\sigma \in \Delta$ of cardinality $i+1$ is called an *i-dimensional face* or an *i-face* of Δ. The empty set, $\emptyset$, is the unique face of dimension -1. Faces of dimension 0, i.e., elements of S, are *vertices* and faces of dimension 1 are *edges*.

The maximal faces under inclusion are called *facets*. To describe a simplicial complex, it is often convenient to simply list its facets—the other faces are exactly determined as subsets. The *dimension* of Δ, denoted $\dim(\Delta)$, is defined to be the maximum of the dimensions of its faces. A simplicial complex is *pure* if each of its facets has dimension $\dim(\Delta)$.

Example 1. If $G = (V, E)$ is a simple connected graph (undirected with no multiple edges or loops), then G is the pure one-dimensional simplicial complex on V with E as its set of facets.

Example 2. Figure 1 pictures a simplicial complex Δ on the set $[5] := \{1,2,3,4,5\}$:

$$\Delta := \{\emptyset, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{12}, \overline{13}, \overline{23}, \overline{24}, \overline{34}, \overline{123}\},$$

writing, for instance, $\overline{23}$ to represent the set $\{2,3\}$.

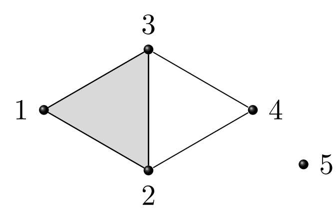


Figure 1. A 2-dimensional simplicial complex, Δ.

The sets of *faces* of each dimension are:

$$F_{-1} = \{\emptyset\} \qquad F_0 = \{\overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$$

$$F_1 = \{\overline{12}, \overline{13}, \overline{23}, \overline{24}, \overline{34}\} \qquad F_2 = \{\overline{123}\}.$$

Its *facets* are $\overline{5}$, $\overline{24}$, $\overline{34}$, and $\overline{123}$. The dimension of Δ is 2, as determined by the facet $\overline{123}$. Since not all of the facets have the same dimension, Δ is not *pure*.

2. Simplicial homology

Let Δ be an arbitrary simplicial complex. By relabeling, if necessary, assume its vertices are $[n] := \{1, \ldots, n\}$. For each i, let $F_i(\Delta)$ be the set of faces of dimension i, and define the *group of i-chains* to be the free abelian group with basis $F_i(\Delta)$:

$$C_i = C_i(\Delta) := \mathbb{Z}F_i(\Delta) := \{\textstyle\sum_{\sigma \in F_i(\Delta)} a_\sigma \, \sigma : a_\sigma \in \mathbb{Z}\}.$$

The *boundary* of $\sigma \in F_i(\Delta)$ is

$$\partial_i(\sigma) := \sum_{j \in \sigma} \mathrm{sign}(j, \sigma) \, (\sigma \setminus j),$$

where $\mathrm{sign}(j, \sigma) = (-1)^{k-1}$ if j is the k-th element of σ when the elements of σ are listed in order, and $\sigma \setminus j := \sigma \setminus \{j\}$. Extending linearly gives the *i-th boundary mapping*,

$$\partial_i \colon C_i(\Delta) \to C_{i-1}(\Delta).$$

If $i > n-1$ or $i < -1$, then $C_i(\Delta) := 0$, and we define $\partial_i := 0$. We sometimes simply write ∂ for ∂_i if the dimension i is clear from context.

Example 3. Suppose $\sigma = \{1, 3, 4\} = \overline{134} \in \Delta$. Then $\sigma \in F_2(\Delta)$, and

$$\mathrm{sign}(1, \sigma) = 1, \quad \mathrm{sign}(3, \sigma) = -1, \quad \mathrm{sign}(4, \sigma) = 1.$$

Therefore,

$$\partial(\sigma) = \partial_2(\overline{134}) = \overline{34} - \overline{14} + \overline{13}.$$

The *(augmented) chain complex* of Δ is the complex

$$0 \longrightarrow C_{n-1}(\Delta) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1(\Delta) \xrightarrow{\partial_1} C_0(\Delta) \xrightarrow{\partial_0} C_{-1}(\Delta) \longrightarrow 0.$$

The word *complex* here refers to the fact that $\partial^2 := \partial \circ \partial = 0$, i.e., for each i, we have $\partial_{i-1} \circ \partial_i = 0$.

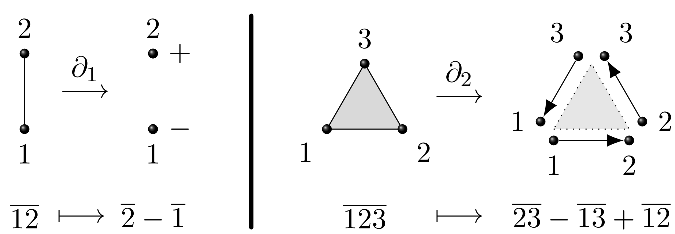


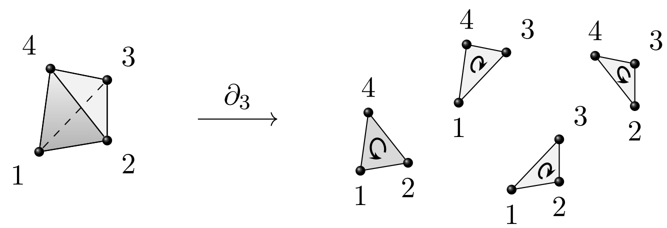
Figure 2. Two boundary mapping examples. Notation: if $i < j$, then we write $i \longrightarrow j$ for $\overline{ij}$ and $i \longleftarrow j$ for $-\overline{ij}$.

Figure 2 gives two examples of the application of a boundary mapping. Note that

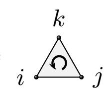
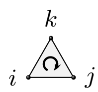
$$\partial^2(\overline{12}) = \partial_0(\partial_1(\overline{12})) = \partial_0(\overline{2} - \overline{1}) = \emptyset - \emptyset = 0.$$

The reader is invited to verify $\partial^2(\overline{123}) = 0$.

Figure 3 shows the boundary of $\sigma = \overline{1234}$, the solid tetrahedron. Figure 4 helps to visualize the fact that $\partial^2(\sigma) = 0$. The orientations of the triangles may be thought of as



$$\overline{1234} \longmapsto \overline{234} - \overline{134} + \overline{124} - \overline{123}$$

Figure 3. ∂_3 for a solid tetrahedron. Notation: if $i < j < k$, then we write  for $\overline{ijk}$ and  for $-\overline{ijk}$.

inducing a "flow" along the edges of the triangles. These flows cancel to give a net flow of 0. This should remind you of Stokes' theorem from multivariable calculus.

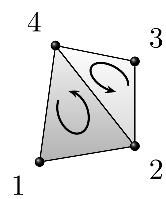


Figure 4. As seen in Figure 3, the boundary of a solid tetrahedron consists of oriented triangular facets.

Example 4. Let Δ be the simplicial complex on $[4]$ with facets $\overline{12}$, $\overline{3}$, and $\overline{4}$ pictured in Figure 5. The faces of each dimension are:

$$F_{-1}(\Delta) = \{\emptyset\}, \quad F_0(\Delta) = \{\overline{1}, \overline{2}, \overline{3}, \overline{4}\}, \quad F_1(\Delta) = \{\overline{12}\}.$$

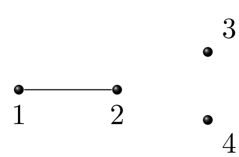


Figure 5. Simplicial complex for Example 4.

Here is the chain complex for Δ:

$$0 \longrightarrow C_1(\Delta) \xrightarrow{\partial_1} C_0(\Delta) \xrightarrow{\partial_0} C_{-1}(\Delta) \longrightarrow 0.$$

$$\overline{12} \longmapsto \overline{2}-\overline{1} \qquad \begin{array}{c}\overline{1}\\ \overline{2}\\ \overline{3}\\ \overline{4}\end{array} \longmapsto \emptyset$$

In terms of matrices, the chain complex is given by

$$0 \longrightarrow \mathbb{Z} \xrightarrow[\partial_1]{\begin{array}{cc} & \overline{12} \\ \begin{array}{c}\overline{1}\\ \overline{2}\\ \overline{3}\\ \overline{4}\end{array} & \begin{pmatrix}-1\\ 1\\ 0\\ 0\end{pmatrix}\end{array}} \mathbb{Z}^4 \xrightarrow[\partial_0]{\begin{array}{cc} & \begin{array}{cccc}\overline{1} & \overline{2} & \overline{3} & \overline{4}\end{array} \\ \emptyset & \begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix}\end{array}} \mathbb{Z} \longrightarrow 0.$$

The sequence is not exact since $\mathrm{rk}(\mathrm{im}\ \partial_1) = \mathrm{rk}\ \partial_1 = 1$, whereas by rank-nullity, $\mathrm{rk}(\ker(\partial_0)) = 4 - \mathrm{rk}\ \partial_0 = 3$.

Definition 5. For $i \in \mathbb{Z}$, the *i-th (reduced) homology* of Δ is the abelian group

$$\widetilde{H}_i(\Delta) := \ker\ \partial_i / \mathrm{im}\ \partial_{i+1}.$$

In particular, $\widetilde{H}_{n-1}(\Delta) = \ker(\partial_{n-1})$, and $\widetilde{H}_i(\Delta) = 0$ for $i > n-1$ or $i < 0$. Elements of $\ker\ \partial_i$ are called *i-cycles* and elements of $\mathrm{im}\ \partial_{i+1}$ are called *i-boundaries*. The *i-th (reduced) Betti number* of Δ is the rank of the i-th homology group:

$$\tilde{\beta}_i(\Delta) := \mathrm{rk}\ \widetilde{H}_i(\Delta) = \mathrm{rk}(\ker \partial_i) - \mathrm{rk}(\partial_{i+1}).$$

Remark 6. To define ordinary (non-reduced) homology groups, $H_i(\Delta)$, and Betti numbers $\beta_i(\Delta)$, modify the chain complex by replacing $C_{-1}(\Delta)$ with 0 and ∂_0 with the zero mapping. The difference between homology and reduced homology is that $H_0(\Delta) \simeq \mathbb{Z} \oplus \widetilde{H}_0(\Delta)$ and, thus, $\beta_0(\Delta) = \tilde{\beta}_0(\Delta) + 1$. All other homology groups and Betti numbers coincide. From now on, we use "homology" to mean reduced homology.

In general, homology can be thought of as a measure of how close the chain complex is to being exact. In particular, $\widetilde{H}_i(\Delta) = 0$ for all i if and only if the chain complex for Δ is exact. For the next several examples, we will explore how exactness relates to the topology of Δ.

The 0-th homology group measures "connectedness". Write $i \sim j$ for vertices i and j in a simplicial complex Δ if $\overline{ij} \in \Delta$. An equivalence class under the transitive closure of $\sim$ is a *connected component* of Δ.

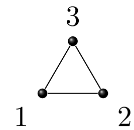
Exercise 7. Show that $\tilde{\beta}_0(\Delta)$ is one less than the number of connected components of Δ.

For instance, for the simplicial complex Δ in Example 4,

$$\tilde{\beta}_0(\Delta) = \mathrm{rk}\ \widetilde{H}_0(\Delta) = \mathrm{rk}(\ker\ \partial_0) - \mathrm{rk}(\partial_1) = 3 - 1 = 2.$$

Example 8. The hollow triangle,

$$\Delta = \{\emptyset, \overline{1}, \overline{2}, \overline{3}, \overline{12}, \overline{13}, \overline{23}\}$$

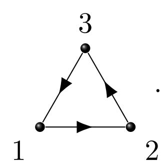


has chain complex

$$0 \longrightarrow \mathbb{Z}^3 \xrightarrow[\partial_1]{\begin{array}{c} \\ \overline{1} \\ \overline{2} \\ \overline{3} \end{array}\begin{array}{c} \begin{array}{ccc} \overline{12} & \overline{13} & \overline{23} \end{array} \\ \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \end{array}} \mathbb{Z}^3 \xrightarrow[\partial_0]{\begin{array}{cc} & \begin{array}{ccc} \overline{1} & \overline{2} & \overline{3} \end{array} \\ \emptyset & \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \end{array}} \mathbb{Z} \longrightarrow 0.$$

It is easy to see that $\operatorname{rk}(\partial_1) = \operatorname{rk}(\ker \partial_0) = 2$. It follows that $\tilde{\beta}_0(\Delta) = 0$, which could have been anticipated since Δ is connected. Since $\operatorname{rk}(\partial_1) = 2$, rank-nullity says $\operatorname{rk}(\ker \partial_1) = 1$, whereas $\partial_2 = 0$. Therefore, $\tilde{\beta}_1(\Delta) = \operatorname{rk}(\ker \partial_1) - \operatorname{rk}(\partial_2) = 1$. In fact, $\widetilde{H}_1(\Delta)$ is generated by the 1-cycle

$$\overline{23} - \overline{13} + \overline{12} =$$



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If we would add $\overline{123}$ to Δ to get a solid triangle, then the above cycle would be a boundary, and there would be no homology in any dimension. Similarly, a solid tetrahedron has no homology, and a hollow tetrahedron has homology only in dimension 2 (of rank 1).

Exercise 9. Compute the Betti numbers for the simplicial complex formed by gluing two (hollow) triangles along an edge. Describe generators for the homology.

Example 10. Consider the simplicial complex pictured in Figure 6 with facets $\overline{14}, \overline{24}, \overline{34}, \overline{123}$. It consists of a solid triangular base whose vertices are connected by edges to the vertex 4. The three triangular walls incident on the base are hollow.

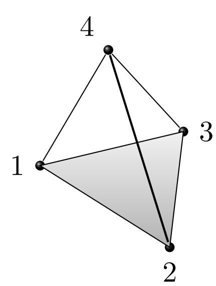
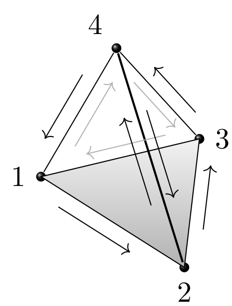


Figure 6. Simplicial complex for Example 10.

What are the Betti numbers? The chain complex is:

$$0 \to \mathbb{Z} \xrightarrow[\partial_2]{\begin{array}{cc} & \overline{123} \\ \begin{array}{c}\overline{12}\\ \overline{13}\\ \overline{14}\\ \overline{23}\\ \overline{24}\\ \overline{34}\end{array} & \begin{pmatrix}1\\-1\\0\\1\\0\\0\end{pmatrix}\end{array}} \mathbb{Z}^6 \xrightarrow[\partial_1]{\begin{array}{cc} & \begin{array}{cccccc}\overline{12} & \overline{13} & \overline{14} & \overline{23} & \overline{24} & \overline{34}\end{array} \\ \begin{array}{c}\overline{1}\\ \overline{2}\\ \overline{3}\\ \overline{4}\end{array} & \begin{pmatrix}-1 & -1 & -1 & 0 & 0 & 0\\ 1 & 0 & 0 & -1 & -1 & 0\\ 0 & 1 & 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 0 & 1 & 1\end{pmatrix}\end{array}} \mathbb{Z}^4 \xrightarrow[\partial_0]{\begin{array}{cc} & \begin{array}{cccc}\overline{1} & \overline{2} & \overline{3} & \overline{4}\end{array} \\ \emptyset & \begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix}\end{array}} \mathbb{Z} \to 0.$$

By inspection, $\mathrm{rk}(\partial_2) = 1$ and $\mathrm{rk}(\partial_1) = \mathrm{rk}(\ker\, \partial_0) = 3$. Rank-nullity gives $\mathrm{rk}(\ker\, \partial_1) = 6 - 3 = 3$. Therefore, $\tilde{\beta}_0 = \tilde{\beta}_2 = 0$ and $\tilde{\beta}_1 = 2$. It is not surprising that $\tilde{\beta}_0 = 0$, since Δ is connected. Also, the fact that $\tilde{\beta}_2 = 0$ is easy to see since $\overline{123}$ is the only face of dimension 2, and its boundary is not zero. Seeing that $\tilde{\beta}_1 = 2$ is a little harder. Given the cycles corresponding to the three hollow triangles incident on vertex 4, one might suppose $\tilde{\beta}_1 = 3$. However, as conveyed in Figure 7, those cycles are not independent: if properly oriented their sum is the boundary of the solid triangle, $\overline{123}$; hence, their sum is 0 in the first homology group.



$$(\overline{12} + \overline{24} - \overline{14}) + (\overline{23} + \overline{34} - \overline{24}) + (\overline{14} - \overline{34} - \overline{13}) = \underbrace{(\overline{12} + \overline{23} - \overline{13})}_{\partial_2(\,\overline{123}\,)}$$

Figure 7. A tetrahedron with solid base and hollow walls. Cycles around the walls sum to the boundary of the base, illustrating a dependence among the cycles in the first homology group.

2.1. A quick aside on algebraic topology. Algebraic topology seeks an assignment of the form $X \mapsto \alpha(X)$ where X is a topological space and $\alpha(X)$ is some algebraic invariant (a group, ring, etc.). If $X \simeq Y$ as topological spaces, i.e., if X and Y are homeomorphic, then we should have $\alpha(X) \simeq \alpha(Y)$ as algebraic objects—this is what it means to be *invariant*. The simplicial homology we have developed provides the tool for creating one such invariant.

Let X be a 2-torus—the surface of a donut. Draw triangles on the surface so that neighboring triangles meet vertex-to-vertex or edge-to-edge. The triangulation is naturally interpreted as a simplicial complex Δ. An amazing fact, of fundamental importance, is that the associated homology groups do not depend on the choice of triangulation! In this way, we get an assignment

$$X \mapsto \widetilde{H}_i(X) := \widetilde{H}_i(\Delta),$$

and, hence, also $X \mapsto \tilde{\beta}_i(X) := \tilde{\beta}_i(\Delta)$, for all i.

In a course on algebraic topology, one learns that these homology groups do not see certain aspects of a space. For instance, they do not change under certain contraction operations. A line segment can be continuously morphed into a single point, and the same goes for a solid triangle or tetrahedron. So these spaces all have the homology of a point—in other words: none at all (all homology groups are trivial). A tree is similarly contractible to a point, so the addition of a tree to a space has no effect on homology. Imagine the tent with missing walls depicted in Figure 6. Contracting the base to a point leaves two vertices connected by three line segments. Contracting one of these line segments produces two loops meeting at a single vertex. No further significant contraction is possible—we are not allowed to contract around "holes" (of any dimension). These two loops account for $\tilde{\beta}_1 = 2$ in our previous calculation. As another example, imagine a hollow tetrahedron. Contracting a facet yields a surface that is essentially a sphere with three longitudinal lines connecting its poles, thus dividing the sphere into 3 regions. Contracting two of these regions results in a sphere—a bubble—with a single vertex drawn on it. No further collapse is possible. This bubble accounts for the fact that $\tilde{\beta}_2 = 1$ is the only nonzero Betti number for the sphere. (Exercise: verify that $\tilde{\beta}_2 = 1$ in this case.)