Math 372 lecture for Friday, Week 9

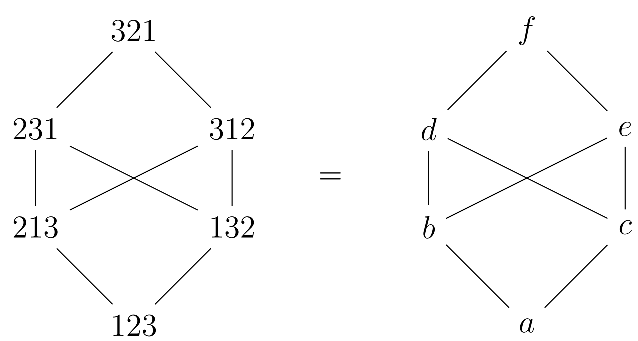
Order complexes

Let P be a finite poset. Recall that a *chain* of length ℓ is a sequence of elements of P of the form $x_0 < x_1 < \cdots < x_\ell$. Create a simplicial complex $\Delta(P)$ corresponding to P by taking the vertices to be the elements of P and the i-faces to be the chains of length i. Note that a subset of a chain is again a chain. The facets of $\Delta(P)$ are the maximal chains of P.

To create the boundary mapping, we need to linearly order the vertices, i.e., the elements of P. We can do this arbitrarily, although we will always do this by refining the given ordering on P. Denote the linear ordering by $\prec$, as opposed to the ordering $<$ on P. Then we will choose $\prec$ so that $x < y$ implies $x \prec y$.

Example. (Bruhat order on $\mathfrak{S}_\mathfrak{n}$). For this example, we will represent each permutation π by the word consisting of the list $\pi(1)\pi(2)\ldots\pi(n)$. For example, if $n = 3$ and $\pi(1) = 2$, $\pi(2) = 3$, and $\pi(3) = 1$, we represent π by 231. An *inversion* of a permutation $\pi \in \mathfrak{S}_n$ is a pair $1 \le i < j \le n$ such that $\pi(i) > \pi(j)$. Define $\ell(\pi)$ to be the number of inversions of π. So for example, $\ell(123) = 0$ and $\ell(321) = 3$, the latter since $3 > 2$, $3 > 1$, and $2 > 1$. The *Bruhat order* on $\mathfrak{S}_n$ has covering relations $\pi \lessdot \tau$ if τ can be obtained from π by a transposition (swapping two elements) and $\ell(\tau) = \ell(\pi) + 1$.

The Hasse diagram for Bruhat order on $\mathfrak{S}_n$:



The order complex for this poset has four facets:

$$\overline{abdf},\ \overline{abef},\ \overline{acdf},\ \overline{acef}.$$

These represent four solid tetrahedra sharing the edge $\overline{af}$ and forming a three dimensional triangulation of a solid sphere. It thus has no homology (i.e., all of the homology groups are 0).

Definition. The *(reduced) Euler characterisic* of a d-dimensional simplicial complex Δ is

$$\widetilde{\chi}(\Delta) := \sum_{i \in \mathbb{Z}} (-1)^i \dim \widetilde{H}_i(\Delta) = \sum_{i=0}^{d} (-1)^i \widetilde{\beta}_i.$$

Exercise. Show that

$$\widetilde{\chi}(\Delta) = -f_{-1} + f_0 - f_1 + \cdots + f_d,$$

where $f_i = |F_i|$, the number of faces of dimension i.

Proposition. (Philip Hall's theorem) Give an finite poset P, let $\widehat{P}$ be the poset formed from P by adjoining $\hat{0}$ and $\hat{1}$ where $\hat{0} < p < \hat{1}$ for all $p \in P$. Let c_i be the number of chains of the form $\hat{0} = p_0 < \cdots < p_i = \hat{1}$. (In particular, $c_0 = 0$ and $c_1 = 1$.) Then

$$\mu_{\hat{P}}(\hat{0}, \hat{1}) = \widetilde{\chi}(\Delta(P)) = c_0 - c_1 + c_2 - \cdots .$$

Proof. Using results from the homework, we have

$$\begin{aligned}
\mu_{\hat{P}}(\hat{0}, \hat{1}) &= \frac{1}{\zeta}(\hat{0}, \hat{1}) \\
&= \frac{1}{\delta + (\zeta - \delta)}(\hat{0}, \hat{1}) \\
&= (\delta - (\zeta - \delta) + (\zeta - \delta)^2 - \cdots)(\hat{0}, \hat{1}) \\
&= \delta(\hat{0}, \hat{1}) - (\zeta - \delta)(\hat{0}, \hat{1}) + (\zeta - \delta)^2(\hat{0}, \hat{1}) - \cdots \\
&= c_0 - c_1 + c_2 - \cdots \\
&= 0 - f_{-1} + f_0 - \cdots \\
&= \widetilde{\chi}(\Delta(P)).
\end{aligned}$$

$\square$

Corollary. Let P be a locally finite poset, and P^* be its *dual poset*, equal to P as a set but with $s < t$ in P^* if $t < s$ in P. (So the Hasse diagram for P^* is the flip of that for P.) Then for all $s < t$ in P, we have

$$\mu_P(s, t) = \mu_{P^*}(t, s).$$

Proof. Let Q be the subposet of P formed from the interval $[s,t]$ by removing s and t, and let Q^* be its dual. Then $\widehat{Q} \simeq [s,t]$, and

$$\mu_{\widehat{Q}}(\hat{0},\hat{1}) = \mu_P(s,t).$$

Similarly

$$\mu_{\widehat{Q^*}}(\hat{0},\hat{1}) = \mu_{P^*}(s,t).$$

However, the c_i's for Q and for Q^* are equal. $\square$