Math 372 lecture for Wednesday, Week 8

Möbius inversion for posets

Let P be a finite poset with Möbius function μ and zeta function ζ. Let K be a field, and recall that $\mathcal{I}(P)$ is the incidence algebra of P whose elements are functions $\xi\colon \mathrm{Int}(P) \to K$, and if $\xi, \eta \in \mathcal{I}(P)$, then

$$(\xi\eta)(x,y) := \sum_{z\in[x,y]} \xi(x,z)\eta(z,y)$$

for all intervals $[x,y] \in \mathrm{Int}(P)$. We now define right and left actions of $\mathcal{I}(P)$ on K^P. If $f \in K^P$, i.e., if $f\colon P \to K$, and $\xi \in \mathcal{I}(P)$, let

$$(f\xi)(t) = \sum_{s:s\leq t} f(s)\xi(s,t)$$

and

$$(\xi f)(t) = \sum_{s:s\geq t} f(s)\xi(t,s),$$

respectively, for all $f \in K^P$, $\xi \in \mathcal{I}(P)$, and $t \in P$.

Theorem. (Möbius inversion) For all $f, g \in K^P$

$$g(t) = \sum_{s:s\leq t} f(s) \quad \forall t \in P \quad\Leftrightarrow\quad f(t) = \sum_{s:s\leq t} \mu(s,t)g(s) \quad \forall t \in P$$

$$g(t) = \sum_{s:s\geq t} f(s) \quad \forall t \in P \quad\Leftrightarrow\quad f(t) = \sum_{s:s\geq t} \mu(t,s)g(s) \quad \forall t \in P.$$

Proof. Since $\mu = \zeta^{-1}$,

$$g = f\zeta \quad\Leftrightarrow\quad f = g\mu$$
$$g = \zeta f \quad\Leftrightarrow\quad f = \mu g.$$

□

The principle of inclusion-exclusion as an instance of Möbius inversion.

Product posets. Given posets P and Q, define their product $P\times Q$ to be the Cartesian product with ordering

$$(p,q) \leq (p',q') \quad\Leftrightarrow\quad p \leq q \text{ and } p' \leq q'.$$

If P and Q are locally finite, then so is $P \times Q$, and we have the following equation for Möbius functions:

$$\mu_{P\times Q}((p,q),(p',q')) = \mu_P(p,p')\mu_Q(q,q').$$

To see this, compute:

$$\begin{aligned}\sum_{(p,q)\le(a,b)\le(p',q')} \mu_P(p,a)\mu_Q(q,b) &= \left(\sum_{p\le a\le p'} \mu_P(p,a)\right)\left(\sum_{q\le b\le q'} \mu_Q(q,b)\right)\\ &= \delta_P(p,p')\delta_Q(q,q')\\ &= \delta_{P\times Q}((p,q),(p',q')).\end{aligned}$$

Möbius function of the boolean poset B_n. Define the poset $\mathbf{2} := \{0,1\}$ with $0 < 1$. Then $\mu(0,0) = \mu(1,1) = 1$, and $\mu(0,1) = -1$. We have an isomorphism of posets

$$\begin{aligned}\mathbf{2}^n = \mathbf{2}\times\cdots\times\mathbf{2} &\xrightarrow{\sim} B_n\\ (a_1,\ldots,a_n) &\mapsto \{i\in[n] : a_i = 1\}.\end{aligned}$$

Let $T \subseteq S \subseteq [n]$, and use the above isomorphism to identify T with $(t_1,\ldots,t_n)$ and S with $(s_1,\ldots,s_n)$ where $t_i, s_i \in \{0,1\}$ for all i. Then,

$$\mu_{B_n}(T,S) = \prod_{i=1}^{n} \mu_{\mathbf{2}}(t_i,s_i) = (-1)^{|S|-|T|}.$$

Principle of inclusion-exclusion (PIE). Letting K be any field, Möbius inversion then says that for all $f, g\colon B_n \to K$

$$g(S) = \sum_{T:T\subseteq S} f(T) \quad \forall S\subseteq[n] \quad\Leftrightarrow\quad f(S) = \sum_{T:T\subseteq S} (-1)^{|S|-|T|} g(T) \quad \forall S\subseteq[n],$$

and dually,

$$g(S) = \sum_{T:T\supseteq S} f(T) \quad \forall S\subseteq[n] \quad\Leftrightarrow\quad f(S) = \sum_{T:T\supseteq S} (-1)^{|T|-|S|} g(T) \quad \forall S\subseteq[n].$$

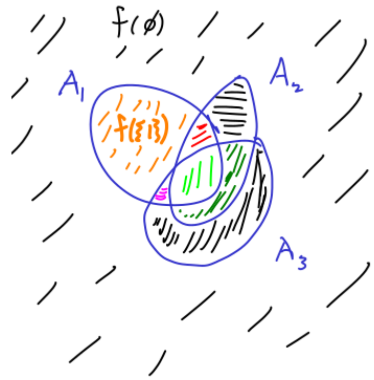
As an instance, suppose that $A_1,\ldots,A_n$ are subsets of some finite set A. Define $f\colon B_n \to \mathbb{Q}$ by

$$f(S) = |\{a\in A : a\in A_i \text{ iff } i\in S\}|.$$

for all $S \subseteq [n]$. Thus, $f(S)$ is the cardinality of the set of elements of A that form a region in the Venn diagram for $A_1, \ldots, A_n$. The "outer" region is counted by

$$f(\emptyset) = |A \setminus \left(\cup_{i\in[n]A_i}\right)| = |A| - |A_1 \cup \cdots \cup A_n| = |A_1^c \cap \cdots \cap A_n^c|.$$

In particular, if $A = \cup_{i\in[n]A_i}$, then $f(\emptyset) = 0$.



Define

$$g(S) = \sum_{T:T\supseteq S} f(T).$$

Then

$$g(S) = \begin{cases} |\cap_{i\in S} A_i| & \text{if } S \neq \emptyset \\ |A| & \text{if } S = \emptyset. \end{cases}$$

Apply the (second version) of Möbius inversion to get

$$\begin{aligned} f(\emptyset) &= \sum_{T:T\supset\emptyset} (-1)^{|T|} g(T) \\ &= |A| - \sum_{i} |A_i| + \sum_{i<j} |A_i \cap A_j| - \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \cdots + (-1)^n |A_1 \cap \cdots \cap A_n|. \end{aligned}$$

On the other hand, we have, $f(\emptyset) = |A| - |A_1 \cup \cdots \cup A_n|$. The **principle of inclusion-exclusion** follows:

$$|A_1 \cup \cdots \cup A_n| = \sum_{i} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|.$$