

Let  $P$  be a poset.

Intervals of  $P$ : If  $x \leq y$  in  $P$ , then

$$[x, y] = \{z \in P : x \leq z \leq y\}.$$

Let  $\text{Int}(P)$  denote the set of all intervals of  $P$ . Note:  $\emptyset \notin \text{Int}(P)$ .

Möbius function of  $P$ :

$$\mu: \text{Int}(P) \rightarrow \mathbb{Z}$$

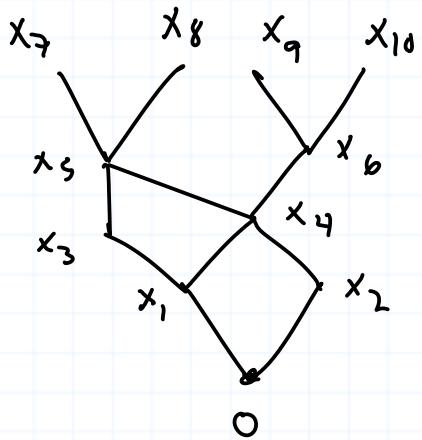
To make sure this sum is finite, we assume  $P$  is locally finite, i.e.,  $[x, y]$  is finite  $\forall x < y$ .

$$\mu(x, x) := 1 \quad \forall x, \quad \mu(x, y) := - \sum_{x \leq z < y} \mu(x, z) \quad \forall x < y.$$

Note: If  $\exists 0 \in P$ , where  $0 \leq x \quad \forall x \in P$ , we define

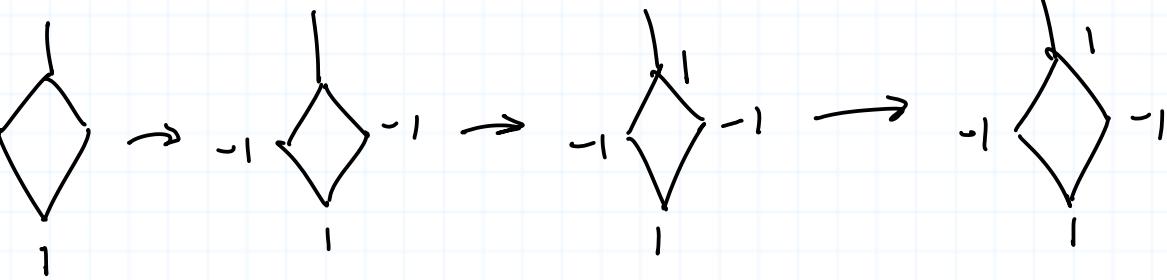
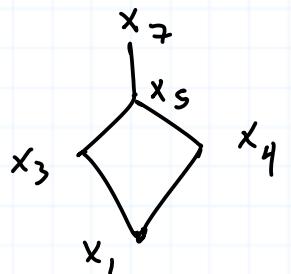
$$\mu(x) = \mu(0, x) \quad \forall x.$$

Example



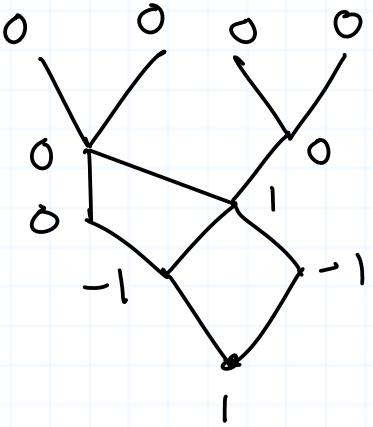
$$[x_1, x_7] = \{x_1, x_3, x_4, x_5, x_7\}$$

To find  $\mu(x_1, x_7)$ :



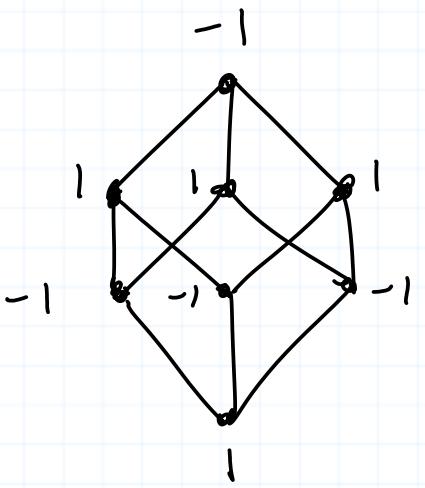
The sum of the number at a vertex with everything below it should be zero at each step.

$$\boxed{\mu(x)}$$



$$\boxed{\mu(x)}$$

$B_3$



## Incidence algebra

Let  $K$  be a field, and let  $P$  be a locally finite poset.

$$\mathbb{J}(P) := \mathbb{J}(P, K) = K^{\text{Int}(P)} = \{ f: \text{Int}(P) \rightarrow K \}.$$

The incidence algebra is a  $K$ -vector space. To make it into a  $K$ -algebra, define the product of  $f, g \in \mathcal{A}(P)$  by

$$(fg)(x,y) := \sum_{x \leq z \leq y} f(x,z)g(z,y).$$

**Identity element:**  $\delta_{(x,y)} = \begin{cases} 1 & \text{if } x=y \\ 0 & \text{otherwise.} \end{cases}$

**Check:**  $(f\delta)(x,y) = \sum_{x \leq z \leq y} f(x,z)\delta(z,y) = f(x,y)\delta(y,y) = f(x,y).$

Similarly,  $\delta f = f$ .  $\square$

**Zeta function of  $P$ :**  $\zeta \in \mathcal{A}(P)$ ,  $\zeta(x,y) = 1 \quad \forall x \leq y$ .

**Prop.**  $\mu = \zeta^{-1}$ , i.e.,  $\mu\zeta = \zeta\mu = \delta$ .

$$\begin{aligned}
 \text{PF/ } (\mu\delta)(x,y) &= \sum_{x \leq z \leq y} \mu(x,z)\delta(z,y) \\
 &= \sum_{x \leq z \leq y} \mu(x,z) = \begin{cases} 1 & \text{if } x=y \\ 0 & \text{if } x \neq y \end{cases}.
 \end{aligned}$$

However, the reverse product is not so straightforward:

$$(\delta\mu)(x,y) = \sum_{x \leq z \leq y} \delta(x,z)\mu(z,y) = \sum_{x \leq z \leq y} \mu(z,y) = ?$$

Instead, the fact that  $\delta\mu = \delta$  is a consequence of the next proposition. In order for the hypotheses to apply we replace  $P$  by the interval  $[x,y]$ .  $\square$

Prop. Let  $R$  be a finite-dimensional  $K$ -algebra with identity  $S$ .

Suppose  $rs = S$  for some  $r, s \in R$ . Then  $sr = S$ .

PF/ The left-multiplication mapping

$$\begin{array}{ccc} R & \xrightarrow{r \cdot} & R \\ t & \longmapsto & rt \end{array}$$

is  $K$ -linear. It is also surjective: Given  $u \in R$ , since  $rs = \delta$ , we have  $r(sr) = (rs)r = \delta r = u$ . Since the domain and codomain have the same finite dimension, the mapping is also injective. Therefore,

$$r(sr) = (rs)r = \delta r = r = r\delta \Rightarrow r(sr - \delta) = 0 \Rightarrow sr - \delta = 0 \Rightarrow sr = \delta.$$

□

**Thm. (Möbius inversion)** Let  $f, g \in K^P$ , i.e.,  $f, g: P \rightarrow K$  where  $P$  is a finite poset. Then,

$$(1) \quad f(x) = \sum_{y: x \leq y} g(y) \quad \forall x \in P \iff g(x) = \sum_{y: x \leq y} \mu(x, y) f(y) \quad \forall x \in P.$$

$$(2) \quad f(x) = \sum_{y: y \leq x} g(y) \quad \forall x \in P \iff g(x) = \sum_{y: y \leq x} \mu(x, y) f(y) \quad \forall x \in P.$$

Example  $P = \mathbb{N}$  or  $\{i, i+1, \dots, n\}$  with usual order.

Möbius:

$$P = \begin{array}{c} n \\ \vdots \\ n-1 \\ \vdots \\ i+1 \\ \vdots \\ i \end{array} \quad = \mu(i, \cdot) \quad \Rightarrow \quad \mu(i, j) = \begin{cases} 1 & \text{if } j = i \\ -1 & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

Discrete fundamental theorem of calculus

$$f(n) = \sum_{i=0}^n g(i) = \sum_{i \leq n} g(i) \Rightarrow g(n) = \sum_{i=0}^n \mu(i, n) f(i) = f(n) - f(n-1), \\ := \Delta f$$