Math 372 lecture for Friday, Week 8

Möbius inversion examples

We first recall a few things from the last lecture.

Möbius inversion. If P is finite and $f, g\colon P \to K$, then

$$g(t) = \sum_{s:s\leq t} f(s) \quad \forall t \in P \quad \Leftrightarrow \quad f(t) = \sum_{s:s\leq t} \mu(s,t)g(s) \quad \forall t \in P$$

$$g(t) = \sum_{s:s\geq t} f(s) \quad \forall t \in P \quad \Leftrightarrow \quad f(t) = \sum_{s:s\geq t} \mu(t,s)g(s) \quad \forall t \in P.$$

Product posets. If P, Q are posets, then $P \times Q$ is a poset with

$$(p,q) \leq (p',q') \quad \Leftrightarrow \quad p \leq q \text{ and } p' \leq q'.$$

and

$$\mu_{P\times Q}((p,q),(p',q')) = \mu_P(p,p')\mu_Q(q,q').$$

Boolean poset. We saw that $B_n \simeq \mathbf{2}^n$ as posets and used the product rule to show

$$\mu_{B_n}(T,S) = (-1)^{|S|-|T|}$$

for all $T \subseteq S \subseteq [n]$.

Principle of inclusion-exclusion. As a special case of Möbius inversion for B_n we showed that if $A_1, \ldots, A_n$ are subsets of some finite set A, then

$$|A_1\cup\cdots\cup A_n| = \sum_i |A_i| - \sum_{i<j} |A_i\cap A_j| + \sum_{i<j<k} |A_i\cap A_j\cap A_k| + \cdots + (-1)^{n-1}|A_1\cap\cdots\cap A_n|.$$

We now give a couple more applications of Möbius inversion.

Derangements revisited. For each $S \subseteq [n]$, let $f(S)$ be the number of elements $\pi \in \mathfrak{S}_n$ whose set of fixed points is exactly S:

$$f(S) = |\,\{\pi \in \mathfrak{S}_n : \pi(i) = i \text{ iff } i \in S\}\,|,$$

and let $g(S)$ be the number of elements $\pi \in \mathfrak{S}_n$ whose set of fixed points includes S:

$$g(S) = |\,\{\pi \in \mathfrak{S}_n : \pi(i) = i \text{ for all } i \in S\}\,|.$$

An easy counting argument and Möbius inversion then gives the number of derangments D_n:

$$\begin{aligned}
D_n &= f(\emptyset) \\
&= \sum_{T:\emptyset\subset T} (-1)^{|T|} g(T) \\
&= \sum_{T:\emptyset\subset T} (-1)^{|T|} (n-|T|)! \\
&= \sum_{i=1}^{n} \sum_{|T|=i} (-1)^{i} (n-i)! \\
&= \sum_{i=1}^{n} (-1)^{i} \binom{n}{i} (n-i)! \\
&= n! \sum_{i=1}^{n} \frac{(-1)^i}{i!}.
\end{aligned}$$

Thus,

$$\frac{D_n}{n!} = \sum_{i=1}^{n} \frac{(-1)^i}{i!} = 1 - 1 + \frac{1}{2} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}.$$

A curiosity. What is the probability that a random function $f\colon [n] \to [n]$ has no fixed points? To answer this, note that to create a such a function, the only restriction is that $f(i) \in [n] \setminus \{i\}$ for each i. Hence, for each i there are $(n-1)$ choices for $f(i)$. This means the total number of such functions is $(n-1)^n$. The total number of functions $[n] \to [n]$ with no restrictions is n^n. Thus, the probability we are looking for is

$$\frac{(n-1)^n}{n^n} = \left(1 - \frac{1}{n}\right)^n.$$

As $n \to \infty$, we have

$$\left(1 - \frac{1}{n}\right)^n \to \frac{1}{e}.$$

Divisibility poset. Let $\mathbb{W} := \mathbb{Z}_{>0} = \{1, 2, \dots\}$, the set of positive integers with a poset structure defined by divisibility: for $a, b \in \mathbb{W}$, let $a \le b$ if $a|b$, i.e., if there exists an integer k such that $b = ka$.

Given $n \in \mathbb{W}$ consider the interval $[1, n]$ as a subposet of $\mathbb{W}$. The Möbius function of $[1, n]$ is equal to the Möbius function of $\mathbb{W}$ restricted to $[1, n]$. Möbius inversion applied to $[1,d] \subset \mathbb{W}$ says that for $f, g\colon [1,d] \to K$,

$$f(n) = \sum_{d:d|n} g(d) \quad \Leftrightarrow \quad g(n) = \sum_{d:d|n} \mu_{\mathbb{W}}(d,n) f(d).$$

Our goal now is to find a formula for $\mu_{\mathbb{W}}$. Factor n into primes: $n = \prod_{i=1}^k p_i^{e_i}$ where each p_i is a prime number and each e_i is a positive integer. Then $a \in [1,n]$ if and only if $a|n$, which is equivalent to saying $a = \prod_{i=1}^k p_i^{f_i}$ where $0 \le f_i \le e_i$ for $i = 1, \dots, k$. Thus, there is an isomorphism of posets

$$\begin{aligned} \mathbb{W} \supset [1,n] &\xrightarrow{\sim} \{0,1,\dots,e_1\} \times \cdots \times \{0,1,\dots,e_k\} \\ a = \prod_{i=1}^k p_i^{f_i} &\mapsto (f_1, \dots, f_k), \end{aligned}$$

where each $\{0,1,\dots,e_i\}$ is a subset of $\mathbb{N}$ with the *usual* ordering of natural numbers. Recall that

$$\mu_{\mathbb{N}}(i,j) = \begin{cases} 1 & \text{if } j = i \\ -1 & \text{if } j = i+1 \\ 0 & \text{if } j > i+1. \end{cases}$$

Suppose that $d = \prod_{i=1}^k p_i^{d_i}$. Using the above isomorphism, we identify $[d,n]$ with a product poset and compute

$$\mu_{\mathbb{W}}(d,n) = \prod_{i=1}^k \mu_{\mathbb{N}}(d_i, e_i) = \begin{cases} 0 & \text{if } e_i - d_i > 1 \text{ for some } i, \\ (-1)^\ell & \text{where } \ell = |\{i : e_i - d_i = 1\}|, \\ 1 & \text{if } e_i = d_i. \end{cases}$$

When we were discussing Dirichlet series, we defined the Möbius function $\mu\colon \mathbb{Z}_{>0} \to \{-1,0,1\}$ as the unique multiplicative function such that

$$\mu(p^e) = \begin{cases} 1 & \text{if } e = 0 \\ -1 & \text{if } e = 1 \\ 0 & \text{if } e > 0 \end{cases}$$

for each prime p and $e \ge 0$. We see that

$$\mu\left(\frac{n}{d}\right) = \mu\left(\prod_{i=1}^k p_i^{e_i - d_i}\right) = \prod_{i=1}^k \mu\left(p_i^{e_i - d_i}\right) = \mu_{\mathbb{W}}(d,n).$$

Möbius inversion on the poset $\mathbb{W}$ thus recovers the classical Möbius inversion formula: if $f(n) = \sum_{d:d|n} g(d)$, then

$$g(n) = \sum_{d:d|n} \mu_W(d,n) f(d) = \sum_{d:d|n} \mu\left(\frac{n}{d}\right) f(d).$$