

Math 372

More generating function examples

Derangements. Let $D_n = \#$ permutations of n objects with no fixed points.

Example S_4 $(1234), (1243), (1324), (1342), (1423), (1432), (12)(34),$
 $(13)(24), (14)(23).$

The number of permutations with exactly k fixed points is then $\binom{n}{k} D_{n-k}$.

Every permutation has some number of fixed points: $n! = \sum_k \binom{n}{k} D_{n-k}$.

$$\text{Then } \{n!\} \xleftrightarrow{e} \sum_n n! \frac{x^n}{n!} = \sum_n x^n = \frac{1}{1-x}.$$

$$\text{and } \left\{ \sum_k \binom{n}{k} D_{n-k} \right\} \sim e^x D \quad \text{where } D \xleftrightarrow{e} \{D_n\}.$$

$$\text{Hence, } \frac{1}{1-x} = e^x D, \quad \text{so} \quad D = \frac{e^{-x}}{1-x}. \quad \text{Therefore,}$$

$$\frac{D_n}{n!} = [x^n] D = [x^n] \frac{e^{-x}}{1-x} = \sum_{k=0}^n (-1)^k \frac{1}{k!}.$$

Hence, the probability of a randomly chosen element of S_n

having no fixed points if $\sum_{k=0}^n (-1)^k \frac{1}{k!}$.

Note that this number tends to $\frac{1}{e}$ as $n \rightarrow \infty$.

Partitions Let $p(n) = \# \text{ partitions of } n$
 $= \# \{ \lambda_1 \geq \dots \geq \lambda_k > 0 : \sum \lambda_i = n, k \in \mathbb{N} \}.$

Hardy, Ramanujan, Rademacher: $p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}$

Let $P = \sum p(n)x^n$. Then

$$P = \prod_{i=1}^{\infty} \frac{1}{1-x^i} = (1+x+x^2+x^3+\dots)(1+x^2+x^{2\cdot 2}+x^{3\cdot 2}+\dots)(1+x^3+x^{2\cdot 3}+x^{3\cdot 3}+\dots)\dots$$

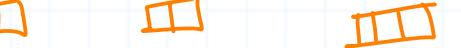
of rows of
size 1
of rows of
size 2
of rows of
size 3
...

Let $p_{\neq}(n) = \# \text{ partitions of } n \text{ with unequal parts}$
 $= \# \{ \lambda_1 > \dots > \lambda_k \geq 0 : \sum \lambda_i = n, k \in \mathbb{N} \},$

and let $P_{\neq} = \sum p_{\neq}(n)x^n$.

Then

$$P_{\neq} = \prod_{i \geq 1} (1+x^i) = (1+x)(1+x^2)(1+x^3)\dots$$



Now note that

$$\begin{aligned} P_{\neq} &= (1+x)(1+x^2)(1+x^3)(1+x^4)\dots \\ &= \frac{1-x^2}{1-x} \cdot \frac{1-x^4}{1-x^2} \cdot \frac{1-x^6}{1-x^3} \cdot \frac{1-x^8}{1-x^4}\dots \\ &= \prod_{i \text{ odd}} \frac{1}{1-x^i} \\ &= \text{generating function for } \# \text{ partitions of } n \text{ into odd parts} = P_{\text{odd}} \end{aligned}$$

Thus, the number of partitions n into unequal parts equals the
number of partition into odd parts. } Due to Euler.

Bijective proof (Glaisher, 1883):

ϕ : partitions into unequal parts \rightarrow partitions into odd parts

Let $\lambda = \lambda_1 > \lambda_2 > \dots > \lambda_k$ and write $\lambda_i = 2^{a_i} \mu_i$ with μ_i odd.

Define $\phi(\lambda) = \mu$ where μ has 2^{a_i} parts of size μ_i .

Example $\lambda = (7, 6, 4, 1) \rightarrow \mu = (\underline{7}, \underline{3}, \underline{3}, \underbrace{1, 1, 1, 1}, \underline{1})$.

$$\begin{array}{ll} 7 = 2^0 \cdot 7 & \text{To go back: } 5 = 2^0 + 2^2 \quad (\text{binary expansion}) \\ 6 = 2^1 \cdot 3 & \\ 4 = 2^2 \cdot 1 & \left. \right\} \text{five } 1\text{s} \\ 1 = 2^0 \cdot 1 & \end{array}$$

Inverse: If μ has k parts of size $2m+1$, write $k = \sum 2^{a_i}$ with a_i distinct (i.e., write k in binary). Then the corresponding λ with unequal parts should include parts of sizes $2^{a_i}(2m+1)$ for each i .

Example of inverse: $n = 8$

odd parts

$$7 + 1 \rightarrow 1 \cdot 7 + 1 \cdot 1 \rightarrow 7 + 1$$

distinct parts

$$5 + 3 \rightarrow 1 \cdot 5 + 1 \cdot 3 \rightarrow 5 + 3$$

$$5 + 1 + 1 + 1 \rightarrow 1 \cdot 5 + (2+1) \cdot 1 \rightarrow 5 + 2 + 1$$

$$3 + 3 + 1 + 1 \rightarrow 2 \cdot 3 + 2 \cdot 1 \rightarrow 6 + 2$$

$$3 + 1 + 1 + 1 + 1 + 1 \rightarrow 1 \cdot 3 + (2^2+1) \cdot 1 \rightarrow 4 + 3 + 1$$

$$1 + 1 + \dots + 1 \rightarrow 2^3 \cdot 1 \rightarrow 8.$$

Note: If $2^{a_i}(2m+1) = 2^{b_i}(2l+1)$ with $a_i \leq b_i$,

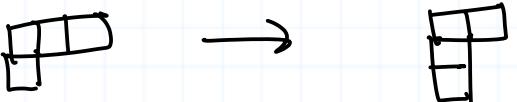
$$\text{we have } 2m+1 = 2^{b_i-a_i}(2l+1) \Rightarrow a_i = b_i, 2m+1 = 2l+1$$

Def. Partition of n is self-conjugate if its corresponding Young diagram is equal to its transpose.

Example



$3+3+2$ is self-conjugate

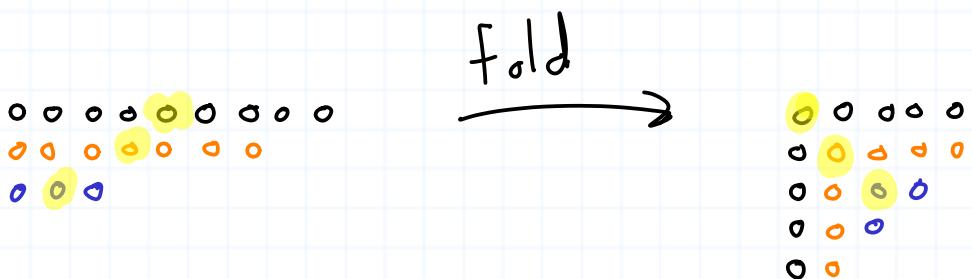


$3+1$ is not.

Prop. The number of partitions of n with odd parts, all distinct, equals the number of self-conjugate partitions of n .

Pf/ Fold each part in the middle to make a right angle: $\circ\circ \dots \circ \color{orange} \circ \color{black} \circ \dots \circ \rightarrow \begin{matrix} \color{orange} \circ & \color{black} \circ \\ \uparrow & \text{middle} \\ \dots & \dots \\ \color{orange} \circ & \color{black} \circ \end{matrix}$

Example $(9, 7, 3) \rightarrow (5, 5, 4, 3, 2)$



□

Dirichlet Series

Take $a_n \in \mathbb{C} \forall n$.

Notation: $f \xleftrightarrow{D} \{a_n\}$ if $\sum_{n=1}^{\infty} \frac{a_n}{n^s} = a_1 + \frac{a_2}{2^s} + \frac{a_3}{3^s} + \dots$ and $s \in \mathbb{C}$.

Say $f \xleftrightarrow{D} \{a_n\}$, $g \xleftrightarrow{D} \{b_n\}$, $h \xleftrightarrow{D} \{c_n\}$. Then

$f + \lambda g \xleftrightarrow{D} \{a_n + \lambda b_n\}$ for $\lambda \in \mathbb{C}$.

$fg \xleftrightarrow{} \left\{ \sum_{d|n} a_d b_{\frac{n}{d}} \right\}$

$$\left(\sum_d \frac{a_d}{d^s} \right) \left(\sum_l \frac{b_l}{l^s} \right) = \sum_{d,l} \frac{a_d b_l}{(d \cdot l)^s} = \sum_n \left(\sum_{d|n} a_d b_{\frac{n}{d}} \right) \frac{1}{n^s}$$

$fgh \xleftrightarrow{} \left\{ \sum_{d_1 d_2 d_3 = n} a_{d_1} b_{d_2} c_{d_3} \right\}$

Examples

$$\zeta(s) := \sum \frac{1}{n^s} \xrightarrow{D} \{1\} \quad \text{Riemann zeta function}$$

$$\zeta^2(s) \xrightarrow{D} \left\{ \sum_{d|n} 1 \cdot 1 \right\} = \{ d(n) \} \quad \text{where } d(n) = \# \text{ divisors of } n.$$

n	1	2	3	4	5	6	7	8	9	10	11	12	...
$d(n)$	1	2	2	3	2	4	2	4	3	4	2	6	

$$\left[\frac{1}{n^s} \right] \zeta^k(s) = \sum_{n_1 \cdots n_k = n} 1$$

$\left[\frac{1}{n^s} \right] (\zeta(s) - 1)^k = \# \text{ ordered factorizations of } n \text{ for which each part is at least 2.}$