

Math 372

Exponential generating functions

$$f \xleftrightarrow{e} \{a_n\} \quad \text{means} \quad f = \sum_{n \geq 0} a_n \frac{x^n}{n!}$$

Say $f \xleftrightarrow{e} \{a_n\}$, $g \xleftrightarrow{e} \{b_n\}$, $h \xleftrightarrow{e} \{c_n\}$. Then

- $f + \lambda g \xleftrightarrow{e} \{a_n + \lambda b_n\}$ for $\lambda = \text{constant}$

- $fg \xleftrightarrow{e} \left\{ \sum_{k=0}^n \binom{n}{k} a_k b_{n-k} \right\}$, $fgh \xleftrightarrow{e} \left\{ \sum_{i+j+k=n} \binom{n}{i j k} a_i b_j c_k \right\}$

$$\left(\sum a_i \frac{x^i}{i!} \right) \left(\sum b_j \frac{x^j}{j!} \right) = \sum_{n \geq 0} \left(\sum_{k=0}^n \frac{1}{k!(n-k)!} a_k b_{n-k} \right) x^n$$

$$= \sum_{n \geq 0} \left(\sum_{k=0}^n \frac{n!}{k!(n-k)!} a_k b_{n-k} \right) \frac{x^n}{n!}$$

$$\binom{n}{i j k} := \frac{n!}{i! j! k!}$$

- $x^k f \xleftrightarrow{e} \left\{ \frac{n!}{(n-k)!} a_{n-k} \right\}$

- $\frac{f - a_0 - a_1 x - \dots - a_{k-1} \frac{x^{k-1}}{(k-1)!}}{x^k} \xleftrightarrow{e} \left\{ \frac{n!}{(n+k)!}, a_{n+k} \right\}$
- $D^k f \xleftrightarrow{e} \{ a_{n+k} \}$
- $x D f \xleftrightarrow{e} \{ n a_n \}$, and if P is a polynomial
 $P(xD)f \xleftrightarrow{e} \{ P(n) a_n \}$

Examples

1. $e^x \xleftrightarrow{e} \{ 1 \} \Rightarrow e^{2x} = e^x \cdot e^x \xleftrightarrow{e} \left\{ \sum_{k=0}^n \binom{n}{k} \right\}$

But $e^{2x} = \sum_n \frac{(2x)^n}{n!} = \sum_n 2^n \frac{x^n}{n!} \xleftrightarrow{e} \{ 2^n \}$.

So $\sum_{k=0}^n \binom{n}{k} = 2^n$ (which is evident from the binomial theorem: $2^n = (1+1)^n = \text{etc.}$)

Applying the same reasoning to e^{3x} gives $3^n = \sum_{i+j+k=n} (i,j,k)$

$$\text{To } e^{kx}: k^n = \sum_{i_1+\dots+i_k=n} (i_1, \dots, i_k) = \sum_{i_1+\dots+i_k=n} \frac{n!}{i_1! \dots i_k!}.$$

2. Let $\{n\}_k$ be the number of partitions of $[n]$ into k nonempty subsets. The $\{n\}_k$ are called the **Stirling numbers of the second kind**.

Example $\{4\}_2 = 7$: $1|234, 2|134, 3|124, 4|123, 12|34, 13|24, 14|23$

$$\text{In general } \{n\}_k = \{n-1\}_{k-1} + k \{n-1\}_k^*$$

(Define $\{0\}_0 = 1$ and

$$\{n\}_0 = 0 \text{ if } n > 0 \text{ and}$$

$$\{n\}_k = 0 \text{ if } n < k \text{ or } n < 0 \text{ or } k < 0.)$$

$$\begin{matrix} & & 1 \\ & 0 & 1 & & \\ 0 & 1 & & 1 \\ & 0 & 1 & 3 & 1 \\ & 0 & 1 & 7 & b & 1 \end{matrix}$$

Proof A partition contains the singleton subset $\{n\}$

or it doesn't. \square

Bell number's $b(n) = \sum_k \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = \# \text{ partitions of } [n].$

Prop. $b(n+1) = \sum_k \binom{n}{k} b(k)$ for $n \geq 0$, and $b(0) = 1$.

Pf/ To form a partition of $[n+1]$ into k parts, first choose k elements of n . Then partition these k elements. Next, group the remaining $n-k$ elements of $[n]$ with $n+1$ in their own pile. \square

Exponential formula for the Bell numbers:

Say $B \xleftrightarrow{e} \{b(n)\}$. Then

$$B' \xleftrightarrow{e} \{b(n+1)\} \stackrel{\text{Prop.}}{=} \left\{ \sum_{k=0}^n \binom{n}{k} b(k) \right\} \xleftrightarrow{e} e^x B.$$

So $B' = e^x B$, $B(0) = 1$. Play around to find the solution:

$$\frac{B'}{B} = e^x \Rightarrow \int \frac{B'}{B} = \int e^x \Rightarrow \log B = e^x + c$$

$$\Rightarrow B = \hat{c} e^{e^x} \quad \text{Then } B(0) = 1 \Rightarrow \hat{c} = e^{-1}. \quad \text{This makes sense}$$

over \mathbb{R} . It is easy to check directly that

3. Fibonacci. Let $F \xleftarrow{e} \{F_n\}$. Then

$$F'' = F' + F \quad \text{with } F_0 = 0, F_1 = 1.$$

$B = e^{e^x - 1}$ satisfies the differential equation (using the chain rule).

Guess a solution of the form $F = e^{rx}$. Then

$$r^2 F = r F + F \Rightarrow r^2 - r - 1 = 0 \Rightarrow \frac{1 \pm \sqrt{5}}{2}. \quad \text{Say } r = \frac{1+\sqrt{5}}{2}, \bar{r} = \frac{1-\sqrt{5}}{2}.$$

Try $F = c_1 e^{rx} + c_2 e^{\bar{r}x}$. Then $F_0 = 0 \Rightarrow c_1 + c_2 = 0$.

and $F_1 = F'(0) = rc_1 + \bar{r}c_2 = 1$. So $c_1(r - \bar{r}) = 1 \Rightarrow c_1 = \frac{1}{\sqrt{5}} = -c_2$.

Now compute $\left[\frac{x^n}{n!}\right] \left(\frac{1}{\sqrt{5}} e^{rx} - \frac{1}{\sqrt{5}} e^{\bar{r}x}\right)$ to get a closed form.