

Math 372

Chapter 7: Enumeration under group action

X a finite set

C any set (of colors)

A coloring of X is a function $f: X \rightarrow C$.

Any subgroup of permutations $G \leq G_X$ acts on the set of colorings by $\pi f := f \circ \pi: X \xrightarrow{\pi} X \xrightarrow{f} C$.

Say colorings f and g are equivalent under G if $\exists \pi \in G$ such that $\pi f = g$.

We are interested in counting inequivalent colorings.

Examples

1. Let $X = [n]$, let $G \leq S_n$, and let $C = \{0,1\}$. Colorings $\{0,1\}^X := \{f: X \rightarrow \{0,1\}\}$ are the same as subsets of $[n]$ via

$f \hookrightarrow \{i : f(i) = 1\}$. For example, if $n=5$, the function



corresponds to the subset $\{2, 4, 5\} \subseteq [n]$.

Let $p_i = \#$ inequivalent colorings with i 1s. Then $p_i = \#\left(\frac{B_n}{G}\right)_i$.

2. $X = \{1, 2, 3, 4, 5\}$ $G = \langle (15)(24) \rangle$

1	2	3	4	5
5	4	3	2	1

G generated by a flip about the middle
(or 180° rotation).

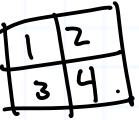
colorings of X with n colors: n^5

colorings up to G -symmetry: Non-symmetrical colorings come in G -equivalent pairs. A symmetrical coloring has the form $\boxed{A B C B A}$.

There are n^3 of these. So up to symmetry, the number of colorings is

$$\frac{1}{2} \left(\underbrace{n^5 - n^3}_{\text{non-symmetric colorings}} \right) + n^3 = \frac{1}{2} (n^5 + n^3)$$

Note: $2 = |G|$.

3. $X = \{1, 2, 3, 4\}$  $C = \{r, b, y\}$

colorings with 2 reds, 1 blue, and 1 yellow:

$$\binom{4}{2} \cdot 2 \cdot 1 = 12$$

choose 2 places for red yellow goes in the remaining spot

choose 1 of the remaining 2 places for b.

$$(i) G = \langle (12)(34) \rangle$$

1	2
4	3

Orbits of G on colorings with 2 reds, 1 blue, and 1 yellow:

r	b
r	y

$$\sim \begin{array}{|c|c|} \hline b & r \\ \hline y & r \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline r & y & \\ \hline r & b & \\ \hline \end{array} \sim \begin{array}{|c|c|c|} \hline y & r & \\ \hline b & r & \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline r & r \\ \hline b & y \\ \hline \end{array} \sim \begin{array}{|c|c|} \hline r & r \\ \hline y & b \\ \hline \end{array},$$

b	y
r	r

$$\begin{array}{|c|c|c|} \hline r & y & \\ \hline b & r & \\ \hline \end{array} \sim \begin{array}{|c|c|c|} \hline y & r & \\ \hline r & b & \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline r & b \\ \hline y & r \\ \hline \end{array} \sim \begin{array}{|c|c|} \hline b & r \\ \hline r & y \\ \hline \end{array}$$

There are 6 inequivalent such colorings.

$$(ii) G = \langle (1243) \rangle \quad 2 \text{ reds, } 1 \text{ blue, } 1 \text{ yellow}$$

r	r
b	y

$$\sim \begin{array}{|c|c|} \hline b & r \\ \hline y & r \\ \hline \end{array} \sim \begin{array}{|c|c|} \hline y & b \\ \hline r & r \\ \hline \end{array} \sim \begin{array}{|c|c|} \hline r & y \\ \hline r & b \\ \hline \end{array}$$

r	b
y	r

$$\sim \begin{array}{|c|c|} \hline y & r \\ \hline r & b \\ \hline \end{array} \sim \begin{array}{|c|c|} \hline r & y \\ \hline b & r \\ \hline \end{array} \sim \begin{array}{|c|c|} \hline b & r \\ \hline r & y \\ \hline \end{array}$$

r	r
y	b

$$\sim \begin{array}{|c|c|} \hline y & r \\ \hline b & r \\ \hline \end{array} \sim \begin{array}{|c|c|} \hline b & y \\ \hline r & r \\ \hline \end{array} \sim \begin{array}{|c|c|} \hline r & b \\ \hline r & y \\ \hline \end{array}$$

3 inequivalent colorings

(iii) $G = \mathbb{G}_4$: All colorings with 2 red, 1 blue, and 1 yellow
are equivalent.

Burnside's Lemma If $G \leq \mathbb{G}_X$ for some finite set X , then

the number of orbits of the action of G on X is

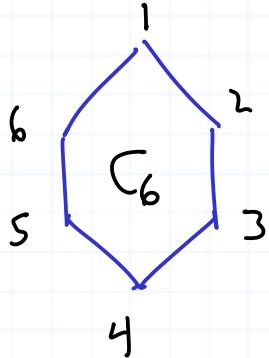
$$\# \left(\frac{X}{G} \right) = \frac{1}{|G|} \sum_{\pi \in G} |\text{Fix}(\pi)|$$

where $\text{Fix}(\pi) = \{x \in X : \pi x = x\}$. So the number of orbits is the average number of fixed points.

Pf/ Math 332 or our text. \square

- * If G acts on X , then it acts on the colorings of X . So *
- * we can use Burnside's lemma to count inequivalent colorings. *

Example Hexagon:



$$\frac{|\pi|}{() \quad \frac{|\text{Fix}(\pi)|}{n^6}}$$

$$G = \langle \sigma \rangle, \quad \sigma = \langle 123456 \rangle.$$

$$C = [n] = \text{colors}$$

$$\sigma = (123456)$$

$$n$$

(only monochromatic colorings are fixed)

$$\sigma^2 = (135)(246)$$

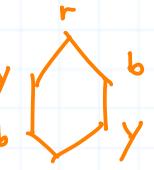
$$n^2$$

(one color for each cycle, (135) and (246))

$$\sigma^3 = (14)(25)(36)$$

$$n^3$$

(for example :)



$$\sigma^4 = (153)(264) \quad n^2$$

$$\sigma^5 = (165432) \quad n$$

$$\# \text{ inequivalent colorings} = \frac{1}{6}(n^6 + n^3 + 2n^2 + 2n)$$

Example $G = \langle (12)(34), (1234) \rangle$ acting on

1	2
4	3

$$\frac{\pi}{\# \text{ Fix}(\pi)}$$

$(1) = (1)(2)(3)(4)$	n^4
$\sigma = (1234)$	n
$\sigma^2 = (13)(24)$	n^2
$\sigma^3 = (1432)$	n
$(12)(34)$	n^2
$(14)(23)$	n^2
$(24) = (1)(24)(3)$	n^3
$(13) = (13)(2)(4)$	n^3

G is the dihedral group, $\# G = 8$: 4 rotations, 4 flips

By Burnside's Lemma, the number of colorings up to G -symmetry is:

$$\frac{1}{8} (n^4 + 2n^3 + 3n^2 + 2n)$$

n	1	2	3	4	5	6
$\#$	1	6	21	55	120	231

Draw representatives for these.

Example Consider $\pi = (148)(2,7,11,5)(6,10,9)$ acting on $[11]$.

How many colorings $f: [11] \rightarrow C = [n]$ are π -fixed?

Answer: n^4 . Make one of n choices for $1,4,8$; one for $2,7,11,5$;
one for $6,10,9$; and one for (3) . $[\pi = (148)(3)(2,7,11,5)(6,10,9)]$

The convention
is to omit
1-cycles.

Thm. Let G be a group of permutations of a finite set.

Then the number of G -inequivalent colorings with n colors is

$$N_G(n) = \frac{1}{\# G} \sum_{\pi \in G} n^{c(\pi)}$$

where $c(\pi)$ is the number of cycles in π .