

## Math 372

More detailed counting

Set up  $G \leq G_X$ ,  $X$  finite set,  $C = \{r_1, r_2, r_3, \dots\}$  colors

$K(i_1, i_2, i_3, \dots) = \#$   $G$ -inequivalent colorings with  $r_k$  used  $i_k$  times.

**Problem:** Compute

$$F_G(r_1, r_2, \dots) = \sum_{i_1, i_2, \dots} K(i_1, i_2, \dots) r_1^{i_1} r_2^{i_2} \dots$$

Polya's solution

For  $\pi \in G_n$ , define  $\text{type}(\pi) = (c_1, c_2, \dots, c_n)$  where  $c_i = \#$  cycles of size  $i$  in  $\pi$ .

**Example**  $\pi = (148)(2,7,11,5)(6,10,9)$  acting on  $[11]$ .

$$\text{type}(\pi) = (1, 0, 2, 1, 0, \dots, 0) \quad \text{type}(1) = (11, 0, 0, \dots)$$

There's an invisible (3) in  $\pi$

## Cycle indicator

For  $\pi$  :  $Z_\pi = z_1^{c_1} \dots z_n^{c_n}$  where  $z_1, \dots, z_n$  are indeterminates

For  $G$  :  $Z_G = Z(z_1, \dots, z_n) = \frac{1}{\#G} \sum_{\pi} Z_\pi$

Then,  $F_G(r_1, r_2, \dots) = Z_G(r_1 + r_2 + \dots, r_1^2 + r_2^2 + \dots, r_1^3 + r_2^3 + \dots, \dots)$

Example  $G =$  dihedral group acting on  $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 4 & 3 \\ \hline \end{array}$  . (13)(2)(4)

|               |           |           |           |           |           |           |             |             | $\pi$                 | $\# \text{Fix}(\pi)$ |
|---------------|-----------|-----------|-----------|-----------|-----------|-----------|-------------|-------------|-----------------------|----------------------|
| $\pi$         | ()        | (1234)    | (13)(24)  | (1432)    | (12)(34)  | (14)(23)  | (24)        | (13)        |                       |                      |
| type( $\pi$ ) | (4,0,0,0) | (0,0,0,1) | (0,2,0,0) | (0,0,0,1) | (0,2,0,0) | (0,2,0,0) | (2,1,0,0)   | (2,1,0,0)   | (1)= (1)(2)(3)(4)     | $n^4$                |
| $Z_\pi$       | $z_1^4$   | $z_4$     | $z_2^2$   | $z_4$     | $z_2^2$   | $z_2^2$   | $z_1^2 z_2$ | $z_1^2 z_2$ | $\sigma = (1234)$     | $n$                  |
|               |           |           |           |           |           |           |             |             | $\sigma^2 = (13)(24)$ | $n^2$                |
|               |           |           |           |           |           |           |             |             | $\sigma^3 = (1432)$   | $n$                  |

$$Z_G = \frac{1}{8} (z_1^4 + 2z_1^2 z_2 + 3z_2^2 + 2z_4)$$

|                                     |       |
|-------------------------------------|-------|
| $\tau = (12)(34)$                   | $n^2$ |
| $\tau \sigma^2 = (14)(23)$          | $n^2$ |
| $\tau \sigma = (24) = (1)(24)(3)$   | $n^3$ |
| $\tau \sigma^3 = (13) = (13)(2)(4)$ | $n^3$ |

$$Z_G = \frac{1}{8} (z_1^4 + 2z_1^2 z_2 + 3z_2^2 + 2z_4)$$

How many colorings up to symmetry with 2 reds, 1 blue, and 1 yellow:

$$F_G(r, b, y) = \frac{1}{8} ((r+b+y)^4 + 2(r+b+y)^2(r^2+b^2+y^2) + 3(r^2+b^2+y^2)^2 + 2(r^4+b^4+y^4))$$

The coefficient of  $r^2by$  in  $(r+b+y)^4$  is  $\frac{4!}{2!1!1!} = 12$

(<sup>★</sup> In general the coefficient of  $x_1^{i_1} \dots x_k^{i_k}$  in  $(x_1 + \dots + x_k)^n$  is the multinomial  $\binom{n}{i_1 \dots i_k} = \frac{n!}{i_1! \dots i_k!}$  where  $i_1 + \dots + i_k = n$ )

The coefficient of  $r^2by$  in  $2(r+b+y)^2(r^2+b^2+y^2)$  is  $2 \cdot \frac{2!}{1!1!} = 4$

The coefficient of  $r^2by$  in  $3(r^2+b^2+y^2)^2$  and  $2(r^4+b^4+y^4)$  is 0.

So  $Z_G(r, b, y) = \frac{1}{8} (\dots + 16r^2by + \dots) = \dots + \underline{\underline{2}}r^2by + \dots$

representatives: 

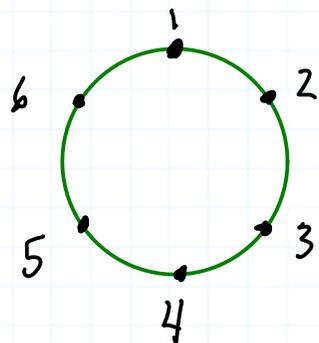
|   |   |
|---|---|
| r | r |
| b | y |

|   |   |
|---|---|
| r | b |
| y | r |

Example Necklaces

$$X = \{1, 2, \dots, \ell\}, \quad G = \langle \sigma \rangle, \quad \sigma = (12 \dots \ell).$$

$$\ell = 6$$



How many colorings with  $n$  colors  $\{r_1, \dots, r_n\}$  up to symmetry?

$$\sigma^0 = ()$$

$$\sigma = (123456)$$

$$\sigma^2 = (135)(246)$$

$$\sigma^3 = (14)(25)(36)$$

$$\sigma^4 = (153)(264)$$

$$\sigma^5 = (15432)$$

In general, letting  $d = \gcd(m, \ell)$ , we have

that  $\sigma^m$  has  $d$  cycles each of length  $\frac{\ell}{d}$ .

So  $\text{type}(\sigma^m) = (0, \dots, d, \dots)$ . The  $\frac{\ell}{d}$ -spot

number of  $m$  with this same type is  $\phi\left(\frac{\ell}{d}\right) = \frac{\ell}{d} \prod_{\substack{p \text{ prime} \\ p | \frac{\ell}{d}}} \left(1 - \frac{1}{p}\right)$  (Euler totient-function)

$$\text{So } Z_G(z_1, \dots, z_\ell) = \sum_m Z_{\pi^m} = \sum_{d | \ell} \phi\left(\frac{\ell}{d}\right) z_{\frac{\ell}{d}}^d = \sum_{d | \ell} \phi(d) z_{\frac{\ell}{d}}^{\frac{\ell}{d}}.$$

$$\text{Therefore, } F_G(r_1, \dots, r_n) = \frac{1}{\ell} \sum_{d | \ell} \phi(d) (r_1^d + \dots + r_n^d)^{\frac{\ell}{d}}. \quad (\text{Say } C = \{r_1, \dots, r_n\}.)$$

Set  $r_1 = r_2 = \dots = r_n = 1$  to get the number of distinct colorings up to symmetry:

$$\frac{1}{l} \sum_{d|l} \phi(d) n^{l/d}.$$

As another special case, consider what happens when there is only one color,  $r$ . Then  $n=1$  and there is only one coloring.

It use  $r$  a total of  $l$  times. So

$$F_G(r) = r^l = \frac{1}{l} \sum_{d|l} \phi(d) r^l. \text{ Which gives the well-known}$$

result

$$\sum_{d|l} \phi(d) = l$$

Example  $l = 6$  :  $\phi(1) + \phi(2) + \phi(3) + \phi(6) = 1 + 1 + 2 + 2 = 6.$