

Math 372

Set-up: $H \subseteq G_n$ subgroup acting on B_n

$P = B_n/H$ quotient poset

If $t = \sum_{x \in B_n} a_x x \in \mathbb{R}B_n$ and $\pi \in G_n$, then $\pi(t) := \sum a_x \pi(x)$.

$\mathbb{R}B_n^H = \{t \in \mathbb{R}B_n : \pi t = t\}$, a graded subspace of $\mathbb{R}B_n$.

For $\mathcal{O} \in P_i$, define $v_{\mathcal{O}} = \sum_{x \in \mathcal{O}} x \in (\mathbb{R}B_n)_i$.

Lemma 5.6. $\{v_{\mathcal{O}}\}_{\mathcal{O} \in P_i}$ is a basis for $(\mathbb{R}B_n^H)_i$.

Pf / See text.

Example $n=4$, $H = \{\overbrace{(1234)}^\sigma\} = \{(), \sigma, \sigma^2, \sigma^3\}$.

basis for $\mathbb{R}B_H$

$$V_{\mathcal{O}_{1234}} = e_{1234}$$

$$V_{\mathcal{O}_{123}} = e_{123} + e_{234} + e_{134} + e_{124}$$

$$V_{\mathcal{O}_{12}} = e_{12} + e_{23} + e_{34} + e_{14}, \quad e_{13} + e_{24} = V_{\mathcal{O}_{13}}$$

$$V_{\mathcal{O}_{12}} = e_1 + e_2 + e_3 + e_4$$

$$V_{\mathcal{O}_1} = e_1 + e_2 + e_3 + e_4$$

$$V_{\mathcal{O}_\emptyset} = e_\emptyset$$

basis for $\mathbb{R}P$

$$\mathcal{O}_{1234}$$

$$\mathcal{O}_{123}$$

$$\mathcal{O}_{12}, \mathcal{O}_{13}$$

$$\mathcal{O}_1$$

$$\mathcal{O}_\emptyset$$

$$\{1234\} = \mathcal{O}_{1234}$$

$$\{123, 234, 134, 124\} = \mathcal{O}_{123}$$

$$\mathcal{O}_{12} = \{12, 23, 34, 14\}$$

$$\{13, 24\} = \mathcal{O}_{13}$$

$$\{1, 2, 3, 4\} = \mathcal{O}_1$$

$$\{\emptyset\} = \mathcal{O}_\emptyset$$

By Lemma 5.6, there are isomorphisms

$$\begin{aligned} \mathbb{R}P_i &\longrightarrow (\mathbb{R}B_n^H)_i \\ \mathcal{O} &\longmapsto \mathcal{V}_\mathcal{O} \end{aligned}$$

for $i = 0, 1, \dots, n$.

Construction of \hat{U}_i Define $\hat{U}_i: \mathbb{R}P_i \rightarrow \mathbb{R}P_{i+1}$ to be the unique mapping making the following diagram commute:

$$\begin{array}{ccc} (\mathbb{R}B_n^H)_i & \xrightarrow{U_i} & (\mathbb{R}B_n^H)_{i+1} \\ \uparrow \cong & & \uparrow \cong \\ \mathbb{R}P_i & \xrightarrow{\hat{U}_i} & \mathbb{R}P_{i+1} \end{array}$$

Claim: \hat{U}_i is an up-operator.

First note that $U_i((\mathbb{R}B_n^H)_i) \subseteq (\mathbb{R}B_n^H)_{i+1}$

Pf/ First we check that $U_i \pi = \pi U_i$ for all $\pi \in H$. Let $x \in (\mathbb{R}B_n)_i$.

Then

$$U_i \pi(x) = \sum_{\substack{\pi(x) \leq y \\ \text{rk}(y) = i+1}} y = \sum_{\substack{x \leq \pi^{-1}y \\ \text{rk}(y) = i+1}} y = \sum_{\substack{x \leq \pi^{-1}y \\ \text{rk}(\pi^{-1}y) = i+1}} y = \sum_{\substack{x \leq z \\ \text{rk}(z) = i+1}} \pi z = \pi U_i(x).$$

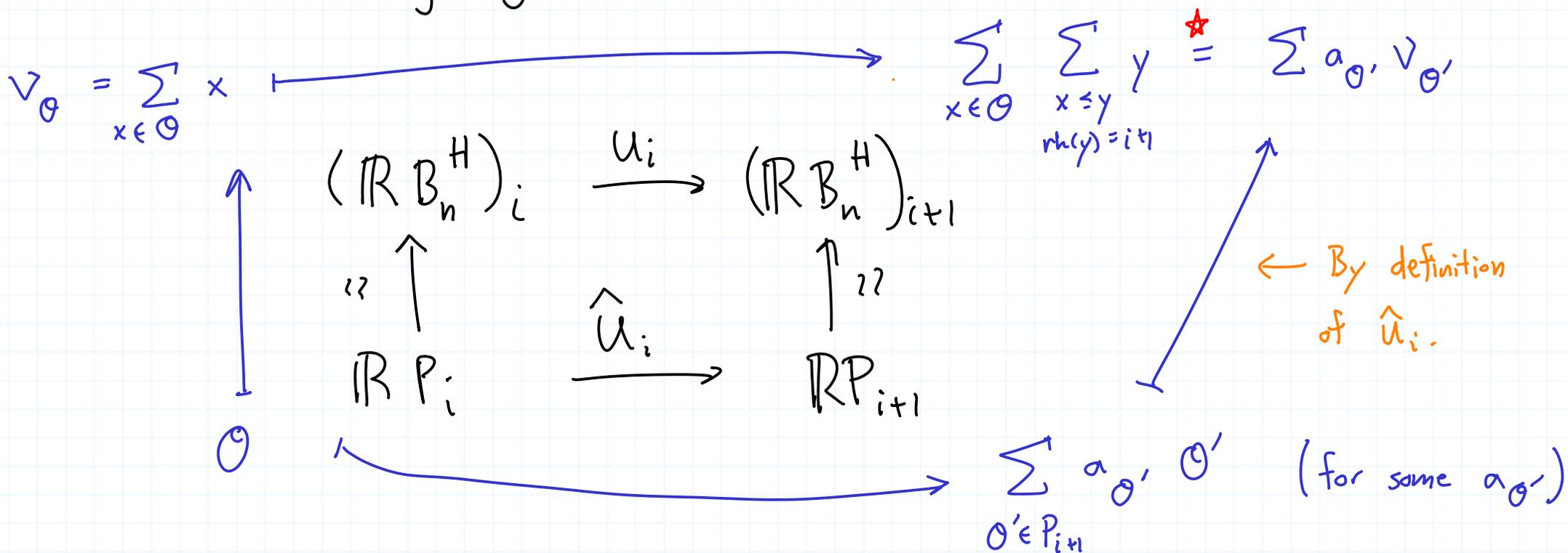
↑ π preserves order
 ↑ letting $z = \pi^{-1}y$

Then, for the basis element $v_\emptyset \in (\mathbb{R}B_n^H)_i$, we have,

$$\pi(U_i v_\emptyset) = U_i(\pi v_\emptyset) = U_i v_\emptyset. \quad \square$$

Now we check \hat{U}_i is an up operator.

Consider the following diagram:



If $a_{\Theta'} \neq 0$, then by \star , there exists $x \in \Theta$ and $y \geq x$ such that $y \in \Theta'$. But this means $\Theta < \Theta'$. Therefore, \hat{u}_i is order raising.

Claim: \hat{U}_i is injective for $i < \frac{n}{2}$

Proof / If $i < \frac{n}{2}$, we've seen $U_i : (\mathbb{R}B_n)_i \rightarrow (\mathbb{R}B_n)_i$ is injective.

Then $U_i : (\mathbb{R}B_n^H)_i \rightarrow (\mathbb{R}B_n^H)_{i+1}$ is just the restriction of the previous mapping to a subspace. So it is still injective. The result follows since the vertical mappings in the commutative diagram are isomorphisms. \square

Claim: \hat{U}_i is surjective for $i \geq \frac{n}{2}$.

Proof / Analogously, we can use the earlier-defined down-operators $D_i : (\mathbb{R}B_n)_i \rightarrow (\mathbb{R}B_n)_{i-1}$ to define down operators $\hat{D}_i : \mathbb{R}P_i \rightarrow \mathbb{R}P_{i-1}$

which are injective for $i > \frac{n}{2}$ and have the property that $\hat{D}_{i+1} = \hat{U}_i^t$.

The result follows since the transpose of an injection is a surjection. \square

As before, it follows that $P = \frac{B_n}{H}$ is Sperner and rank unimodal.

Corollary. B_n/H is rank-symmetric, unimodal, and Sperner.

Proof/ Rank symmetry follows by taking complements of sets. The unimodal and Sperner property follows from our order-raising operators \hat{U}_i . \square

Corollary. The poset of unlabeled graphs on n vertices is rank-symmetric, unimodal and Sperner.