Math 372 lecture for Friday, Week 4

Definition. A sequence of real numbers $a_0, \ldots, a_n$ is *logarithmically concave* or *log-concave* is

$$a_i^2 \geq a_{i-1}a_{i+1}$$

for $1 \leq i \leq n-1$. We say $a_0, \ldots, a_n$ is *strongly log-concave* if

$$b_i^2 \geq b_{i-1}b_{i+1}$$

for $1 \leq i \leq n-1$ where $b_i := a_i/\binom{n}{i}$.

The name *log-concave* comes from the fact that $a_i^2 \geq a_{i-1}a_{i+1}$ is a multiplicative version of the inequality $a_i \geq \frac{a_{i-1}+a_{i+1}}{2}$. An easy calculation shows that strong log-concavity is equivalent to

$$a_i^2 \geq \left(1+\frac{1}{i}\right)\left(1+\frac{1}{n-i}\right)a^{i-1}a^{i+1},$$

from which it follows that strong log-concavity implies log-concavity.

It is not always true that log-concavity implies unimodality. Consider $1, 0, 0, 1$, for instance. A sequence, $a_0, a_1, \ldots, a_n$ has *no internal zeros* if whenever $i < j < k$ and both a_i and a_k are nonzero, then a_j is also nonzero.

Proposition 5.11 Let $\alpha = (a_0, \ldots, a_n)$ be a sequence of nonnegative real numbers with no internal zeros. If α is log-concave, the α is unimodal.

Proof. If at most two values of α are nonzero, then $a_{i-1}a_{i-1} = 0$ for all $1 \leq i \leq n-1$, and the result holds. Otherwise, assume α is not unimodal. Then there exists $1 \leq i \leq n-1$ such that $a_{i-1} > a_i < a_{i+1}$. Using the fact that the elements of α are nonnegative, it follows that $a_i^2 < a_{i-1}a_{i+1}$. □

Example. The n-th row of Pascal's triangle

$$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$$

is trivially strongly log-concave with no internal zeros. Hence, it is unimodal. Here is a combinatorial proof of log-concavity of the binomial coefficients from Richard Stanley's paper Log-concave and unimodal sequences in algebra, combinatorics, and geometry, 1989 (see the "Final word of warning" at the end of that paper). Let $\binom{[n]}{i}$ denote the collection of subsets of $[n]$ of size i. We describe an injection

$$\phi\colon \binom{[n]}{i-1} \times \binom{[n]}{i+1} \to \binom{[n]}{i} \times \binom{[n]}{i}$$

Log-concavity then follows:

$$a_{i-1}a_{i+1} = \left|\binom{[n]}{i-1}\right|\left|\binom{[n]}{i+1}\right| \leq \left|\binom{[n]}{i}\right|^2 = a_i^2.$$

For any $X \subseteq [n]$ and $j = 0, \ldots, n$, define

$$X_j = X \cap [j] = \{x \in X : x \leq j\}.$$

Given $(A, B) \in \binom{[n]}{i-1} \times \binom{[n]}{i+1}$, let j be the smallest nonnegative integer such that $|A_j| = |B_j| - 1$. This integer must exist since $|A_0| = |B_0| = 0$, and $|A_n| = |B_n| - 2$, and the sizes of A_j and B_j grow in size by at most one each time j increases by one. Next, define

$$C := A_j \cup (B \setminus B_j) \quad \text{and} \quad D := B_j \cup (A \setminus A_j).$$

Note that these are disjoint unions since, for instance $A_j \subseteq [j]$ and $B \setminus B_j \subset [n] \setminus [j]$. We also have that $|C| = |D| = i$: instance,

$$|C| = |A_j| + |B| - |B_j| = |A_j| + (i+1) - (|A_j| + 1) = i,$$

and similarly for $|D|$. Define $\phi(A, B) = (C, D)$. For injectivity, note that

$$A_k = C_k \quad \text{and} \quad B_k = D_k$$

for $k = 0, \ldots, j$, while

$$A_{j+1} \neq C_{j+1} \quad \text{and} \quad B_{j+1} \neq D_{j+1}.$$

Thus, we can recover j, A_j, and B_j from (C, D). But from that information, we can recover (A, B). For instance,

$$A = C_j \cup (D \setminus B_j).$$

That completes the proof.

Example. Let $A = \{1, 3, 4, 6\}$. Then $A_0 = \emptyset$, $A_1 = A_2 = \{1\}$, $A_3 = \{1, 3\}$, $A_4 = A_5 = \{1, 3, 4\}$, and $A_6 = A$.

There is a surprising connection between log-concavity and polynomials with real zeros:

Theorem 5.12 (I. Newton) Let

$$P(x) = \sum_{i=0}^{n} b_i x^i = \sum_{i=0}^{n} \binom{n}{i} a_i x^i$$

be a real polynomial all of whose zeros are real numbers. Then the sequence $b_0, \ldots, b_n$ is strongly log-concave, or equivalently, the sequence $a_0, \ldots, a_n$ is log-concave. Moreover, if each $b_i \geq 0$ (so the zeros of $P(x)$ are non-positive [why?]) then the sequence $b_0, \ldots, b_n$ has no internal zeros.

Note that Stanley's statement here reverses the notation introduced earlier: in the definition of strong log-concavity, we had $b_i = a_i / \binom{n}{i}$, and were talking about strong log-concavity of the sequence (a_i) and here we have $a_i = b_i / \binom{n}{i}$ and are talking about strong log-concavity of the sequence (b_i).

Before proving the theorem, we first prove some preliminary results.

Proposition. Let $F(x) = \sum_{i=0}^{m} a_i x^i$ be a real polynomial of degree m with only real zeros. Then its derivative, $F'(x)$, has only real zeros.

Proof. By long division, α is a zero of F of multiplicity $r \geq 1$ if and only if there exists a polynomial G such that

$$F(x) = (x - \alpha)^r G(x)$$

where $G(\alpha) \neq 0$. So suppose α is a zero of F of multiplicity r and choose G as above. Then,

$$F'(x) = r(x - \alpha)^{r-1} G(x) + (x - \alpha)^r G'(x) = (x - \alpha)^{r-1}(G(x) + (x - \alpha)G'(x)).$$

Since $G(x) + (x - \alpha)G'(x)$ evaluated at α is $G(\alpha) \neq 0$, we conclude that α is a zero of F' of multiplicity $r - 1$.

Say the zeros of F are $\alpha_1 < \ldots < \alpha_k$ with respective multiplicities $r_1, \ldots, r_k$. By Rolle's theorem, for each $1 \leq i < k$, since $F(\alpha_i) = F(\alpha_{i+1}) = 0$, there is zero β_i of $F'(x)$ between α_i and α_{i+1}. Thus, we have zeros for $F'(x)$:

$$\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_{k-1} < \beta_{k-1} < \alpha_k.$$

where α_i has multiplicity $r_i - 1$. Counting multiplicities, we have found $(\sum_{i=1}^{k} r_i) - 1 = \deg(F) - 1 = m - 1$ roots of F'. Since $\deg(F') = m - 1$, we have found all of the roots of F', and they are all real. $\square$

Proposition. Let $F(x) = \sum_{i=0}^{m} a_i x^i$ be a real polynomial of degree m with only real zeros. Then the polynomial $\widetilde{F}(x) := x^m F(1/x)$ has only real zeros.

Proof. Let $k \geq 0$ be the multiplicity of 0 as a zero of F. We can then write $F(x) = x^k G(x)$ where $G(0) \neq 0$. The nonzero zeros of F and G are the same, with the same multiplicities. We have

$$\widetilde{F}(x) = x^{m-k} G(1/x).$$

So the nonzero zeros of $\widetilde{F}$ are the reciprocals of those of F. Thus, all of the zeros of $\widetilde{F}$ are real. $\square$

The above two propositions say that the property of having all real zeros is preserved under taking derivatives and under a certain type of "inversion". Here is an example illustrating how these operations are useful in relating the property of having all real zeros with strong log-concavity. Say

$$P(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 x^5 + b_6 x^6 + b_7 x^7$$

has all real zeros. We can show that strong log-concavity holds for, say, b_2, b_3, b_4 by using the two operations from the propositions to eliminate all the terms in P except the three involving b_2, b_3 and b_4. First take two derivatives to eliminate b_0 and b_1:

$$P''(x) = (2 \cdot 1)\, b_2 + (3 \cdot 2)\, b_3 x + (4 \cdot 3)\, b_4 x^2 + (5 \cdot 4)\, b_5 x^3 + (6 \cdot 5)\, b_6 x^4 + (7 \cdot 6)\, b_7 x^5.$$

Now invert to define a polynomial Q:

$$Q(x) = x^5 P''(1/x) = (2{\cdot}1)\, b_2 x^5 + (3{\cdot}2)\, b_3 x^4 + (4{\cdot}3)\, b_4 x^3 + (5{\cdot}4)\, b_5 x^2 + (6{\cdot}5)\, b_6 x + (7{\cdot}6)\, b_7.$$

Take 3 derivatives to eliminate the terms involving b_5, b_6, and b_7:

$$Q^{(3)} = (2 \cdot 1)(5 \cdot 4 \cdot 3)\, b_2 x^2 + (3 \cdot 2)(4 \cdot 3 \cdot 2)\, b_3 x + (4 \cdot 3)(3 \cdot 2 \cdot 1)\, b_4.$$

To prove strong log-concavity, define $a_i = b_i / \binom{7}{i}$ and substitute:

$$\begin{aligned} Q^{(3)} &= (2 \cdot 1)(5 \cdot 4 \cdot 3) \binom{7}{2} a_2 x^2 + (3 \cdot 2)(4 \cdot 3 \cdot 2) \binom{7}{3} a_3 x + (4 \cdot 3)(3 \cdot 2 \cdot 1) \binom{7}{4} a_4 \\ &= (2 \cdot 1)(5 \cdot 4 \cdot 3) \frac{7!}{2!5!} a_2 x^2 + (3 \cdot 2)(4 \cdot 3 \cdot 2) \frac{7!}{3!4!} a_3 x + (4 \cdot 3)(3 \cdot 2 \cdot 1) \frac{7!}{4!3!} a_4 \\ &= \frac{7!}{2} (a_2 x^2 + 2 a_3 x + a_4) \end{aligned}$$

Since $Q^{(3)}$ has only real zeros, it must be that the quadratic

$$a_2 x^2 + 2 a_3 x + a_4$$

has only real zeros. By the quadratic equation, this happens if and only if the discriminant is nonnegative:

$$4a_3^2 - 4a_2 a_4 = 4(a_3^2 - a_2 a_4) \geq 0.$$

That gives log-concavity for a_2, a_3, a_4, and hence strong log-concavity for b_2, b_3, b_4.

Proof of Theorem 5.12. Given P as in the statement of the theorem, suppose that $\deg(P) = m$ (allowing for the fact that $b_{m+1}, \ldots, b_n$ could all be 0). Fix $1 < i < m$, and take derivatives and inversions to eliminate all terms in P except that involving b_{i-1}, b_i, and b_{i+1}. The result is the following polynomial which also has only real zeros:

$$\frac{m!}{2}(a_{i-1}x^2 + 2a_i x + a_{i+1}).$$

Therefore, the sequence of b_i is strongly log-concave.

Next suppose that each $b_i \geq 0$. For the sake of contradict, suppose there are internal zeros. Then there exists $i + 1 < k$ where b_i and b_j are positive and $b_j = 0$ for all $i < j < k$. By taking derivatives and inverting, as above, eliminate the terms of P having degree less then i or larger then k. We are left with a polynomial having two terms (since $b_j = 0$ for all $i < j < k$):

$$c + dx^{k-i}$$

for some positive c, d, and with $k - i \geq 2$. Further, this polynomial has only real zeros. To find them we solve

$$x^{k-i} = -\frac{c}{d}.$$

This equation has $k - i$ solutions over the complex numbers, with at most one real solution (when $k - i$ is odd, then $-\sqrt[k-i]{c/d}$ is a solution). Thus, we reach a contradiction. $\square$

Answer to the "why?" in Theorem 5.12: If P is constant, then 0 is the only possible zero of P. Otherwise, since the $b_i \geq 0$, the derivative $P'(x)$ is positive for $x > 0$. Since $P(0) = b_0 \geq 0$, it follows that $P(x) > 0$ for $x > 0$.