Math 372 lecture for Monday, Week 3

$P = B_n$, boolean poset

$$\text{order-raising operators:} \quad \begin{aligned} U_i \colon \mathbb{R}P_i &\to \mathbb{R}P_{i+1} \\ x &\mapsto \textstyle\sum_{y \in P_{i+1} : x < y} y \end{aligned}$$

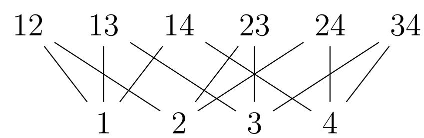
$$\text{order-lowering operators:} \quad \begin{aligned} D_i \colon \mathbb{R}P_i &\to \mathbb{R}P_{i-1} \\ y &\mapsto \textstyle\sum_{x \in P_{i-1} : x < y} x \end{aligned}$$

Goal:

Theorem 4.7. U_i is injective for $i < \frac{n}{2}$ and surjective for $n \geq \frac{n}{2}$.

It then follows from last time that B_n is Sperner.

Example. $n = 4$



$$U_1 \colon \mathbb{R}^4 \xrightarrow{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}} \mathbb{R}^6 \qquad D_2 \colon \mathbb{R}^6 \xrightarrow{\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}} \mathbb{R}^4$$

In general, $D_{i+1} = U_i^t$.

Lemma 4.6. $D_{i+1}U_i - U_{i-1}D_i = (n - 2i)I$.

Proof. See text. $\square$

Proof of Theorem 4.7. If A is any real matrix, then AA^t is real and symmetric, hence, diagonalizable. We also have that for all v

$$v^t AA^t v = (A^t v)^t (A^t v) \geq 0,$$

i.e., AA^t is positive semidefinite, which implies all of its eigenvalues are nonnegative real numbers. By Lemma 4.6,

$$D_{i+1}U_i - U_{i-1}D_i = (n-2i)I.$$

Say $v_1, \dots, v_{p_i}$ are the eigenvectors of $U_{i-1}D_i = U_{i-1}U_{i-1}^t$ with corresponding eigenvectors $\lambda_k \geq 0$. Then

$$\begin{aligned}(D_{i+1}U_i)v_k &= (U_{i-1}D_i)v_k + (n-2i)v_k \\ &= \lambda_k v_k + (n-2i)v_k \\ &= (\lambda_k + n - 2i)v_k.\end{aligned}$$

Hence, the eigenvalues of $D_{i+1}U_i$ are $\{\lambda_k + n - 2i\}$ for $k = 1, \dots, p_i$. If $i < \frac{n}{2}$, the $\lambda_k + n - 2i > 0$ for all k, and since $D_{i+1}U_i$ thus has no zero eigenvalues, it is invertible. That implies U_i is injective for $i < \frac{n}{2}$.

For the case $i \geq \frac{n}{2}$, use Lemma 4.6 again but in the form

$$U_i D_{i+1} = D_{i+2}U_{i+1} + (2i + 2 - n)I.$$

If $\mu_1, \dots, \mu_{p_{i+1}}$ are the eigenvalues of $D_{i+2}U_{i+1}$ then $\mu_k \geq 0$ for all k, and the eigenvalues of $U_i D_{i+1}$ are $\{\mu_k + 2i + 2 - n\}$, which are all positive for $i \geq \frac{n}{2}$. Hence, $U_i D_{i+1}$ is invertible. It follows that U_i is surjective. $\square$