Math 372 lecture for Wednesday, Week 2

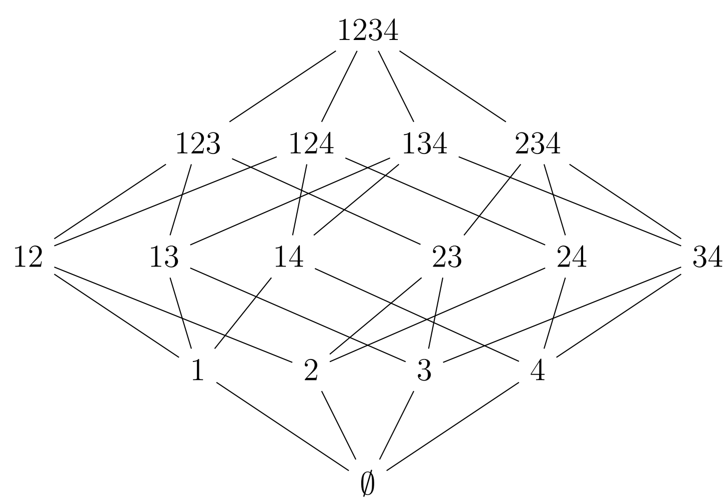
Definition. A *poset* (*partially ordered set*) is a finite set P and a relation $\leq$ defined for some (not necessarily all) elements of P satisfying:

1. $x \leq x$ for all $x \in P$ (reflexivity)
2. $x \leq y$ and $y \leq x$ imply $x = y$ (antisymmetry)
3. $x \leq y$ and $y \leq z$ imply $x \leq z$ (transitivity).

Given $\leq$, we use the usual conventions for the symbols $\geq$, $<$, $>$, $\lneq$, $\not\leq$ etc. If neither $x \leq y$ nor $y \leq x$, we say x and y are *incomparable*.

Example. Let X be a finite set, and let $P = 2^X$, the collection of all subsets of X. For $U, V \in P$ define $U \leq V$ if $U \subseteq V$ in X. Then $(P, \leq)$ is a poset called a *boolean poset* and denoted B_X. If $X = [n] := \{1, \ldots, n\}$, then we write B_n for B_X.

If P is a poset and $x, y \in P$, we say y *covers* x and write $x \lessdot y$ if $x < y$ and there does not exist $z \in P$ such that $x < z < y$. The *Hasse diagram* for a poset is a graph in the plane whose vertices are the elements of P, the element x is drawn below y if $x < y$, and there is an edge between x and y if y covers x. Here is the Hasse diagram for B_4:



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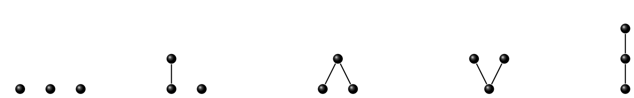
Example. There are five posets with three elements:



Question. What is the size of the largest collection of incomparable elements in B_n?

So we are looking for the largest collection of subsets of $[n]$ such that if U and V are in the collection, then neither $U \subseteq V$ nor $V \subseteq U$. It turns out that for B_4, the answer is

$$\{12, 13, 14, 23, 24, 34\}.$$

In preparation for answering this question in general, we review some vocabulary from the text (Chapter 4). Let P be a poset. Then a string of elements of P of the form

$$x_0 \leq x_1 \leq \cdots \leq x_n$$

is called a *chain* in P. A chain of the form

$$x_0 < x_1 < \cdots < x_n$$

has *length* n. A chain is *maximal* if it is not contained in a larger chain. If every maximal chain has length n, then we say P is *graded of rank* n. A chain is *saturated* if it has the form

$$x_0 \lessdot x_1 \lessdot \cdots \lessdot x_i$$

for some i. If P is graded of rank n and $x \in P$, then the *rank* of x is j, written $\rho(x) = j$ if j is the length of the largest saturated chain in P having top element x. The i-th *level* of P is

$$P_i = \{x \in P : \rho(x) = i\}.$$

Let $p_i := |P_i|$. Then the rank generating function for the graded poset P is

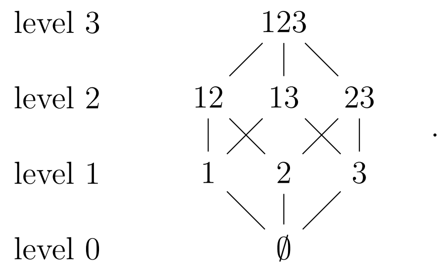
$$F(P, q) = \sum_{i=0}^{n} p_i q^i.$$

We say P is *rank-symmetric* if $p_i = p_{n-i}$ for $0 \leq i \leq n$, and *rank unimodal* if

$$p_0 \leq p_1 \leq \cdots \leq p_j \geq \cdots \geq p_{n-1} \geq p_n$$

for some j.

Example. The boolean poset B_3 is graded of rank 3 and has 6 maximal chains. The levels are:



The sizes of the levels are $p_0 = 1$, $p_1 = 3$, $p_2 = 3$, and $p_3 = 1$. This sequence, $1, 3, 3, 1$ is symmetric and unimodal. The rank generating function is

$$F(B_3, q) = 1 + 3q + 3q^2 + q^3.$$

In general, B_n is rank-symmetric and unimodal with $p_i = \binom{n}{i}$ (since P_i is the number of subsets of $[n]$ of cardinality i) and rank generating function

$$F(B_n, q) = \sum_{i=0}^{n} \binom{n}{i} q^i = (1 + q)^n.$$

An *antichain* in any poset P is a subset of P in which no distinct pair of elements is comparable. For instance, in B_3, the following sets are antichains:

$$\{2\}\,, \quad \{1, 3\}\,, \quad \{12, 34\}\,, \quad \{1, 23, 24\}\,.$$

The following is then a rewording of our question:

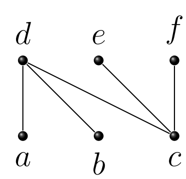
Question. What is the size of the largest antichain in B_n?

Definition. A graded poset P of rank n has the *Sperner property* if the size of the largest antichain in P is the maximal cardinality of a level of P:

$$\max\{|A| : A \text{ an antichain of } P\} = \max\{|P_i| : i = 0, \ldots, n\}\,.$$

Note that P_i is always an antichain.

Non-example. The following is the Hasse diagram for a graded poset of rank 2 that does not have the Sperner property:



There is one largest antichain, $\{a, b, e, f\}$, and it has size 4, whereas $|P_0| = |P_1| = 3$.

Exercise. Draw the Hasse diagrams of all posets of size 3 having the Sperner property.