Math 372 lecture for Friday, Week 2

The goal for the next two class periods is to develop general tools which we will apply to see that the boolean poset B_n is Sperner.

Let P be a graded poset of rank n. For each $0 \leq i \leq n$, let P_i be the i-th level of P, and let $p_i = |P_i|$. An *order matching from P_i to P_{i+1}* is a function

$$\mu\colon P_i \hookrightarrow P_{i+1}$$

such that $x < \mu(x)$ for all $x \in P_i$. An *order matching from P_{i+1} to P_i* is a function

$$\mu\colon P_{i+1} \hookrightarrow P_i$$

such that $\mu(x) < x$ for all $x \in P_{i+1}$.

Proposition 4.4. Suppose there exists $0 \leq j \leq n$ and order matchings

$$P_0 \xhookrightarrow{\mu_0} P_i \xhookrightarrow{\mu_2} \cdots \xhookrightarrow{\mu_{j-1}} P_j \xhookleftarrow{\mu_{j+1}} P_{j+1} \xhookleftarrow{\mu_{j+2}} \cdots \xhookleftarrow{\mu_n} P_n.$$

Then P is unimodal and Sperner.

Proof. If an order matching exists, since the μ_i are injections, then

$$p_0 \leq p_1 \leq \cdots \leq p_j \geq p_{j+1} \geq \cdots \geq p_n,$$

i.e., P is unimodal. Recall the Sperner property:

$$\max\{|A| : A \text{ an antichain of } P\} = \max\{p_i : 0 \leq i \leq n\}.$$

To see that P is Sperner:

1. In the Hasse diagram for P, for each $i \neq j$, color the edge connecting x to $\mu_i(x)$ red for all $x \in P_i$.
2. Let G be the graph whose vertices are the elements of P, and whose edges are the edges in the Hasse diagram that are colored red. Some of the vertices in P_j may be isolated. Color these isolated vertices red.
3. The components of G partition P into disjoint chains (considering each isolated (red) vertex as a chain of length 0). Each of these chains meets P_j in a unique point. So the number of chains is $|P_j|$.
4. Let A be an antichain in P. Then: (i) each $a \in A$ is in one of the above chains, and (ii) each of the above chains contains at most one element of A. Thus, the number of chains above is at least $|A|$.

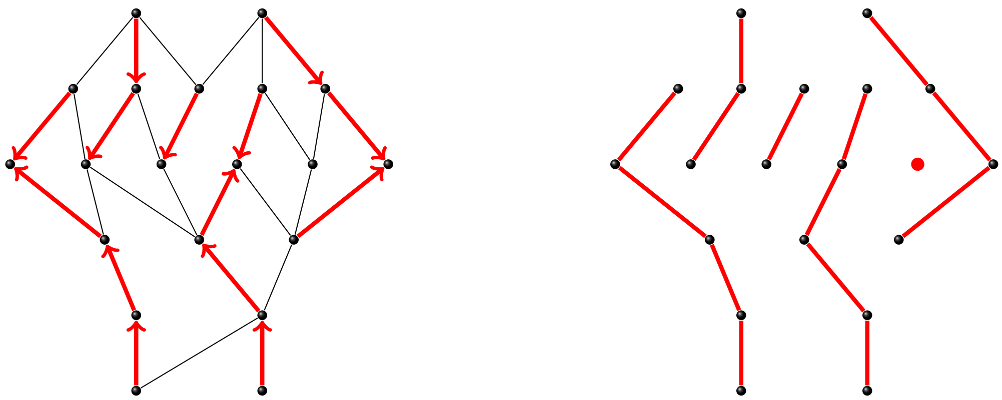

Figure 1: Order matchings and chains

5. Done.

See Figure 1 for an example. $\square$

Question. How should we go about finding order relations as in Proposition 4.4?

And *order-raising operator* is a linear function $U \colon \mathbb{R}P_i \to \mathbb{R}P_{i+1}$ such that for all $x \in P_i$, we have that $U(x)$ is a linear combination of elements $y \in P_{i+1}$ such that $x \leq y$.

Lemma 4.5. Suppose $U \colon \mathbb{R}P_i \to \mathbb{R}P_{i+1}$ is an injective order-raising operator. Then there exists an order matching $P_i \hookrightarrow P_{i+1}$. Similarly, suppose $U \colon \mathbb{R}P_i \to \mathbb{R}P_{i+1}$ is a surjective order-raising operator. Then there exists order matching $P_{i+1} \hookrightarrow P_i$.

Proof.

1. Consider the matrix A representing U. Its columns are are indexed by P_i and its rows are indexed by P_{i+1}.

$$\begin{array}{cc} & \begin{array}{cccc} x_1 & x_2 & \cdots & x_{p_i} \end{array} \\ \begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_{p_i} \\ \vdots \\ y_{p_{i+1}} \end{array} & \left[\begin{array}{c} \\ \\ \star \\ \\ \\ \\ \end{array} \right] \end{array}$$

$$\mathbb{R}^{p_i} \longrightarrow \mathbb{R}^{p_{i+1}}$$

2. Since U is injective, the columns of A are linearly independent. So A has rank p_i. Since column-rank equals row-rank, this means there is a selection of p_i linearly independent rows of A. Throwing out the other rows, we arrive at a square submatrix A' of A with rank p_i indexed by all of P_i and a subset of P_{i+1} of size p_i.

$$A' = \begin{array}{c} \\ y_{j_1} \\ y_{j_2} \\ \vdots \\ y_{j_{p_i}} \end{array} \begin{array}{c} x_1 \; x_2 \cdots x_{p_i} \\ \left[\begin{array}{ccc} & & \\ & \bigstar & \\ & & \end{array} \right] \end{array}$$

3. We have $0 \neq \det(A') = \sum_\pi \operatorname{sgn}(\pi) a'_{1\pi(1)} \cdots a'_{p_i\pi(p_i)}$, summing over permutations π of $[p_i] := \{1, \cdots, p_i\}$. In particular, there exists a π for which the corresponding summand is nonzero. Think of this π as a *rook placement* in the matrix A', i.e., a selection of entries in each row such that no two are in the same column. The entries in this rook placement are nonzero. Let e be one of these entries of A'. Then e is in a row indexed by an element $y \in P_{i+1}$ and is in a column indexed by an element x of P_i. We define the order matching by letting $x \mapsto y$. For example, we might have

$$\begin{array}{c} \\ y_{j_1} \\ y_{j_2} \\ y_{j_3} \end{array} \begin{array}{c} \;\; x_1 \quad x_2 \quad x_3 \\ \begin{bmatrix} * & * & \otimes \\ \otimes & * & * \\ * & \otimes & * \end{bmatrix} \end{array}$$

$$x_1 \mapsto y_{j_1}, \quad x_2 \mapsto y_{j_2}, \quad x_1 \mapsto y_{j_3}.$$

□