Math 372 lecture for Wednesday, Week 1

Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, \ldots, v_p\}$ and edge set E. Let A be the adjacency matrix for G. Since A is real and symmetric, there exist real linearly independent orthonormal eigenvectors $u_1, \ldots, u_p$ for A. Say the corresponding eigenvalues are $\lambda_1, \ldots, \lambda_p$. Last time, we saw that:

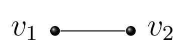
- The number of ℓ-walks (i.e., walks of lenght ℓ) from v_i to v_j is

$$\sum_{k=1}^{p} u_{ik} u_{jk} \lambda_k^{\ell}.$$

- The number of closed ℓ-walks is

$$\operatorname{tr}(A^{\ell}) = \sum_{k=1}^{p} \lambda_k^{\ell}.$$

Example 1. Here is a toy example:



$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By inspection, $(1, 1)$ and $(1, -1)$ are orthogonal eigenvectors with corresponding eigenvalues 1 and -1. So we can take

$$u_1 = \frac{1}{\sqrt{2}}(1, 1), \quad \lambda_1 = 1$$

$$u_2 = \frac{1}{\sqrt{2}}(1, -1), \quad \lambda_2 = -1.$$

The number of closed ℓ-walks is

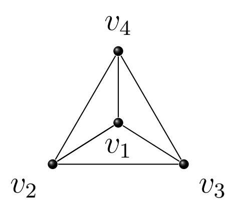
$$1^{\ell} + (-1)^{\ell} = \begin{cases} 2 & \text{if } \ell \text{ is even} \\ 0 & \text{if } \ell \text{ is odd.} \end{cases}$$

The number of ℓ-walks from v_1 to v_2 is

$$\begin{aligned}
\sum_{k=1}^{2} u_{1k}u_{2k}\lambda_k^\ell &= u_{11}u_{21}\lambda_1^\ell + u_{12}u_{22}\lambda_2^\ell \\
&= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} 1^\ell + \frac{1}{\sqrt{2}} \cdot \frac{-1}{\sqrt{2}} (-1)^\ell \\
&= \frac{1}{2}\left(1 - (-1)^\ell\right) \\
&= \begin{cases} 1 & \text{if } \ell \text{ is odd} \\ 0 & \text{if } \ell \text{ is even.} \end{cases}
\end{aligned}$$

Both of these results are clear by looking at the graph, without any calculation.

Example 2. (Complete graphs) Let's apply our theory to the complete graph K_p on p vertices. This graph has an edge joining each pair of (distinct) vertices. For example, in the case $p = 4$, we have the following:



$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

ℓ	# ℓ-walks v_1 to v_1	# closed ℓ-walks
0	1	4
1	0	0
2	3	12
3	$3 \cdot 2 \cdot 1 = 6$	24
4	?	?

In general, the adjacency matrix for K_p is

$$A = J - I$$

where J is the $p \times p$ matrix with $J_{ij} = 1$ for all i, j, and I is the $p \times p$ identity matrix.

Lemma. The eigenvalues for J are 0 (multiplicity $p - 1$) and p.

Proof. Letting e_i denote the i-th standard basis vector for $\mathbb{R}^p$, we have that

$$\{e_i - e_p : i = 1, \dots, p-1\}$$

are eigenvectors for J with eigenvalue 0, and $e_1 + \cdots + e_p$ is an eigenvector with eigenvalue p. $\square$

Proposition. The eigenvalues for K_p are -1 (multiplicity $p-1$) and $p-1$.

Proof. By the lemma, the characteristic polynomial for J is

$$p_J(x) = \det(J - Ix) = \prod_{i=1}^{p} (\lambda_i - x) = \pm x^{p-1}(x-p).$$

The characteristic polynomial for $A = J - I$ is

$$\begin{aligned} p_A(x) &= \det(A - Ix) = \det(J - I - Ix) = \det(J - I(x+1)) \\ &= p_J(x+1) = \pm(x+1)^{p-1}(x+1-p) \\ &= \pm(x+1)^{p-1}(x-(p-1)). \end{aligned}$$

The result follows. $\square$

As an immediate consequence, we get:

Corollary.

1. The number of closed ℓ-walks in K_p is

$$(p-1)(-1)^\ell + (p-1)^\ell.$$

2. For each i, the number of closed ℓ-walks from v_i to itself is

$$\frac{1}{p}\left((p-1)(-1)^\ell + (p-1)^\ell\right).$$

The second part follows by symmetry: the number of closed ℓ-walks at each vertex is the same.

Proposition. The number of ℓ-walks in K_p from v_a to v_b when $a \neq b$ is

$$\frac{1}{p}\left((p-1)^\ell - (-1)^\ell\right).$$

Proof. One may do the count by directly computing the i, j-entry of A^ℓ as shown in the text using the binomial theorem. We will give another argument (hinted at in the text). First note that the total number of ℓ-walks is $p(p-1)^\ell$. (There are p starting points, and from each vertex reached in a walk, there are $p-1$ choices for an edge along which the walk could continue.) Using the previous corollary and symmetry, we have

$$\begin{aligned} p(p-1)^\ell &= \sum_{i,j=1}^{p} (A^\ell)_{ij} \\ &= \sum_{i=1}^{p} (A^\ell)_{ii} + \sum_{i\neq j} (A^\ell_{ij}) \\ &= p\left(\frac{1}{p}\left((p-1)(-1)^\ell + (p-1)^\ell\right)\right) + p(p-1)\tau \end{aligned}$$

where τ is the number of walks from v_i to v_j for any fixed $i \neq j$. (The last summand has this form since there are p choices for vertex v_i and $p-1$ choices for vertex v_j, and the number of walks between them is the same for all $i \neq j$.) Solving for τ gives the formula we are looking for. □

Combinatorial proof. By the previous corollary, the number of closed walks of length $\ell + 1$ starting at v_a is

$$\frac{1}{p}\left((p-1)(-1)^{\ell+1} + (p-1)^{\ell+1}\right).$$

Each of these walks is in bijective correspondence with a walk of length ℓ starting at v_a: stop at the penultimate vertex of the walk, just before returning to v_a. There are $p-1$ choices for the penultimate vertex, and the number of walks of length ℓ from v_a to any vertex $v_b \neq v_a$ is independent of the $p-1$ choices for b. So fixing a particular b, the number of walks from v_a to v_b is

$$\frac{1}{p-1}\left(\frac{1}{p}\left((p-1)(-1)^{\ell+1} + (p-1)^{\ell+1}\right)\right),$$

and the result follows. □