Math 372 lecture for Monday, Week 1

For administrative details, see our course homepage:

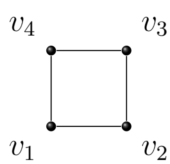
http://people.reed.edu/~davidp/372/.

Topic I. Walks in Graphs.

Question. How many walks are there of length ℓ between two given vertices of a graph G? (The *length* is the number of edges traversed.)

Note. Throughout today's lecture, G will be a simple graph with vertices $v_1, \dots, v_p$.

Example. Consider the question on $G = C_4$, the cycle graph on four vertices;



length	walks from v_1 to v_3
0	—
1	—
2	$v_1v_2v_3$, $v_1v_4v_3$
3	—
4	$v_1v_2v_1v_2v_3$, $v_1v_2v_1v_4v_3$, $v_1v_2v_3v_2v_3$
	$v_1v_2v_3v_4v_3$, $v_1v_4v_1v_2v_3$, $v_1v_4v_1v_4v_3$
	$v_1v_4v_3v_2v_3$, $v_1v_4v_3v_4v_3$

Definition. The adjacency matrix for G is the $p \times p$ matrix A given by

$$A_{ij} = \begin{cases} 1 & \text{if } G \text{ has an edge from } v_i \text{ to } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition. Let A be the adjacency matrix for G. Then the number of walks of length ℓ in G from v_i to v_j is $(A^\ell)_{ij}$.

Proof. We prove this by induction. The base case, $\ell = 0$, is trivial. For $\ell \geq 1$, say $A^{\ell-1} = B = (b_{mn})$. Then

$$(A^\ell)_{ij} = (BA)_{ij} = \sum_{k=1}^{p} B_{ik}A_{kj}.$$

By induction,

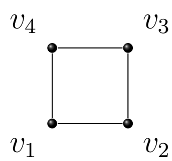
$$b_{ik} = \#\text{ walks from } v_i \text{ to } v_k \text{ of length } \ell - 1.$$

We also have

$$a_{kj} = \begin{cases} 1 & \text{if } G \text{ has an edge from } v_k \text{ to } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

Each walk from v_i to v_j of length at least one must pass through some vertex v_k to v_j in its final step, and the final edge must be $\{v_k, v_j\}$. The result follows. □

Example. Consider $G = C_4$, again. Here are some powers of the adjacency matrix:



$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \qquad A^2 = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix}, \qquad A^3 = \begin{pmatrix} 0 & 4 & 0 & 4 \\ 4 & 0 & 4 & 0 \\ 0 & 4 & 0 & 4 \\ 4 & 0 & 4 & 0 \end{pmatrix},$$

$$A^4 = \begin{pmatrix} 8 & 0 & 8 & 0 \\ 0 & 8 & 0 & 8 \\ 8 & 0 & 8 & 0 \\ 0 & 8 & 0 & 8 \end{pmatrix}.$$

Compare the $1,3$-entries in the above matrix with walks displayed in the previous example.

Theorem. A real $p \times p$ symmetric matrix A has p orthonormal eigenvectors, i.e., there exist $u_1, \ldots, u_p \in \mathbb{R}^p$ with

$$u_i \cdot u_j = \delta(i,j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

and there exist (not necessarily distinct) $\lambda_1, \ldots, \lambda_p \in \mathbb{R}$ such that

$$Au_i = \lambda_i u_i$$

for $i = 1, \ldots, p$.

Proof. Linear algebra. □

Corollary. Let A be the adjacency matrix for G with (not necessarily distinct) eigenvalues $\lambda_1, \ldots, \lambda_p$. Then the number of walks of length ℓ from v_i to v_j is

$$\sum_{k=1}^{p} u_{ik} u_{jk} \lambda_k^{\ell}$$

where $u_q := (u_{1q}, u_{2q}, \ldots, u_{pq})$ for $q = 1, \ldots, p$.

Proof. With notation as in the Theorem, let U be the $p \times p$ matrix whose columns are the u_i. Then $U^t U = I_p$, so $U^{-1} = U^t$, and

$$U^{-1} A U = \operatorname{diag}(\lambda_1, \ldots, \lambda_p) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_p \end{pmatrix}.$$

Hence,

$$(U^{-1}AU)^{\ell} = (U^{-1}AU)(U^{-1}AU) \cdots (U^{-1}AU) = U^{-1} A^{\ell} U,$$

but also

$$(U^{-1}AU)^{\ell} = \operatorname{diag}(\lambda_1, \ldots, \lambda_p)^{\ell} = \operatorname{diag}(\lambda_1^{\ell}, \ldots, \lambda_p^{\ell}).$$

Letting $D = (d_{st}) = \operatorname{diag}(\lambda_1^{\ell}, \ldots, \lambda_p^{\ell})$, it follows that $A^{\ell} = U D^{\ell} U^{-1}$. Therefore,

$$\begin{aligned} (A^{\ell})_{ij} &= \sum_{k=1}^{p} u_{ik} (D^{\ell} U^{-1})_{kj} \\ &= \sum_{k=1}^{p} u_{ik} (D^{\ell} U^{t})_{kj} \\ &= \sum_{k=1}^{p} u_{ik} \left(\sum_{s=1}^{p} (D^{\ell})_{ks} u_{js} \right) \\ &= \sum_{k=1}^{p} u_{ik} \lambda_k^{\ell} u_{jk}. \end{aligned}$$

□

Corollary. The number of closed walks of length ℓ in G, i.e., the number of walks of length ℓ beginning and ending at the same vertex, is $\sum_{k=1}^{p} \lambda_k^{\ell}$.

Proof. From our Proposition, the number of closed walks of length ℓ is the sum of the diagonal entries of A^{ℓ}, i.e., $\operatorname{tr}(A^{\ell})$, the *trace* of A^{ℓ}. By a standard theorem from linear algebra, the trace of a square matrix is the sum of its eigenvalues. Now note that

$$A^{\ell}u_i = A^{\ell-1}(\lambda_i u) = \lambda_i A^{\ell-1}u_i = \cdots = \lambda_i^{\ell}u_i$$

for $i = 1, \ldots, p$. So the eigenvalues for A^{ℓ} are λ_i^{ℓ} for $i = 1, \ldots, p$. The result follows. □

Example. Letting $G = C_4$ be the cycle graph from the first example, we have

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

The eigenvalues for A, with multiplicities, are $0, 0, 2, -2$. So the number of closed walks of length ℓ for this cycle graph is

$$0^{\ell} + 0^{\ell} + 2^{\ell} + (-2)^{\ell}.$$

Note that $(0)^0 = 1$.