Math 372 lecture for Friday, Week 1

Let G be any simple graph with vertices $v_1, \dots, v_p$, and let E be the set of edges. Suppose we have a weight function for the edges

$$\mathrm{wt}\colon E \to R$$

where R is any commutative ring. Let A_{wt} be the *weighted adjacency matrix* defined by $(A_{\mathrm{wt}})_{ij} = \mathrm{wt}(v_i, v_j)$, where we define $\mathrm{wt}(v_i, v_j) = 0$ if v_i, v_j is not an edge. Given any walk $\mathbf{w}$ in G, define the weight of $\mathbf{w}$ to be the product of the weights of its edges. Then a straightforward generalization of the argument used during the first day of class shows that $(A_{\mathrm{wt}}^{\ell})_{ij}$ is the sum of the weights of all ℓ-walks starting at v_i and ending at v_j, Again, the proof goes by induction, the case $\ell = 0$ being trivial. Let $\ell \geq 1$, and let $B := A_{\mathrm{wt}}^{\ell}$. Then

$$(A_{\mathrm{wt}}^{\ell})_{ij} = (BA_{\mathrm{wt}})_{ij} = \sum_{k=1}^{p} B_{ik}(A_{\mathrm{wt}})_{kj} = \sum_{k=1}^{p} B_{ik}\,\mathrm{wt}(v_k, v_j)$$

By induction, B_{ik} is the sum of the weights of all walks from v_i to v_k, and the result follows.

To recover the original result, take $R = \mathbb{Z}$ and all weights equal to 1. We have that A_{wt} is just the usual adjacency matrix, and $(A^{\ell})_{ij}$ is the number of ℓ-walks from v_i to v_j.

Next let G be a connected *multigraph* with vertex set V. This means there may be multiple edges connecting the same two vertices. For each $u, v \in V$, let μ_{uv} be the number of edges joining u to v, and let $d_u = \deg(u)$ be the *degree* of u, i.e., the number of edges incident on u.

A *random walk* on G is one in which having arrived at a vertex u in the walk, the next edge is determined uniformly at random among the edges incident on u. Thus, one would move from u to v with probability $\mu(u, v)/d_u$. Let $M = M(G)$ be the *probability matrix* with rows and columns indexed by V. The u, v-th entry of M is defined to be

$$M_{uv} := \frac{\mu(u, v)}{d_u}.$$

Lemma. The probability of a random walk starting at u ends at v is $(M^{\ell})_{uv}$.

Proof. Let G' be the graph G where each multiedge is replaced by a single edge, then M may be interpreted as a weighted adjacency matrix A_{wt} for G'. □ □

So, as earlier, the question of diagonalizability becomes important. Luckily:

Theorem 3.2. The probability matrix M is diagonalizable and has only real eigenvalues.

Proof. Since G is connected, $d_u > 0$ for all $u \in V$. Let D be the diagonal matrix with rows indexed V and with v-th diagonal entry $\sqrt{d_v}$ for all $v \in V$. A calculation shows

$$(DMD^{-1})_{uv} = \frac{\mu_{uv}}{\sqrt{d_u d_v}}$$

for all $u, v \in V$. Hence, DMD^{-1} is symmetric and real, hence, diagonalizable with real eigenvalues. Since M is similar to DMD^{-1}, it is also diagonalizable and has the same eigenvalues. $\square$ $\square$

Hitting times.

Let $S(u, v)$ consisting of all walks from u to v meeting v for the first time at the last step of the walk. Then consider the random variable with sample space $S(u, v)$ defined by

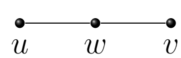
$$\begin{aligned} X \colon S(u,v) &\to \mathbb{R} \\ \mathbf{w} &\mapsto \text{length}(\mathbf{w}). \end{aligned}$$

For each $u, v \in V$, define the *hitting time* to be the expected length of a walk starting at u and ending the first time it meets v:

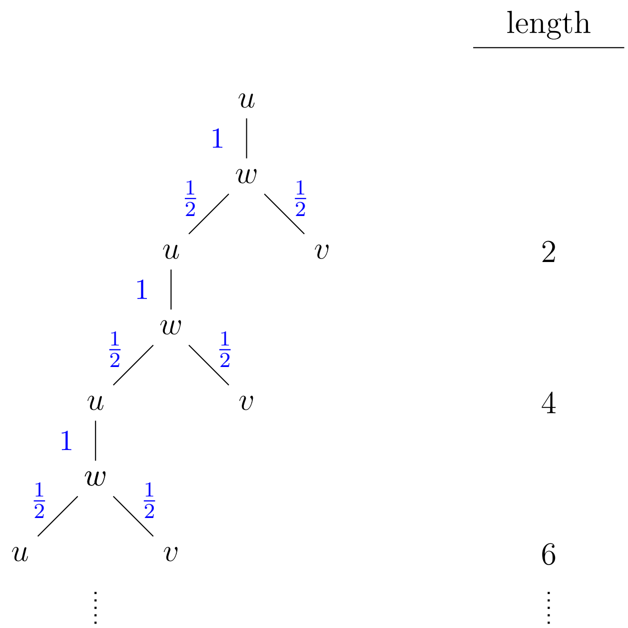
$$H(u,v) := \mathbb{E}(X) := \sum_{n \geq 0} n p_n$$

where $p_n := \mathbb{P}(X = n)$, the probability the length of the random walk is n.

Example. $G = P_3$, the path graph with 3 vertices:



We can calculate the hitting time $H(u, v)$ using the following tree (the blue labels denote the probabilities):



The hitting time is

$$
\begin{aligned}
H(u,v) &= \tfrac{1}{2}\cdot 2 + \left(\tfrac{1}{2}\right)^2 \cdot 4 + \left(\tfrac{1}{2}\right)^3 \cdot 6 + \dots \\
&= 1 + \tfrac{1}{2}\cdot 2 + \left(\tfrac{1}{2}\right)^2 \cdot 3 + \dots \\
&= \sum_{n\geq 0} (n+1)\left(\tfrac{1}{2}\right)^n .
\end{aligned}
$$

Note that $\sum_{n\geq 0} z^{n+1} = \frac{z}{1-z}$ for $|z|<1$ (geometric series). Differentiating, we get that

$$\sum_{n\geq 0} (n+1) z^n = \frac{1}{(1-z)^2}$$

also converges for $|z|<1$. Therefore, $H(u,v)=4$.
Alternatively, if Y is the random variable

$$
\begin{aligned}
Y \colon S(u,v) &\to \mathbb{R} \\
\mathbf{w} &\mapsto \begin{cases} 0 & \text{if length}(\mathbf{w}) \leq 2 \\ 1 & \text{otherwise,} \end{cases}
\end{aligned}
$$

then we may use the fact that

$$\mathbb{E}(X) = \mathbb{E}(X|Y=0)\mathbb{P}(Y=0) + \mathbb{E}(X|Y=1)\mathbb{P}(Y=1)$$

to get

$$H(u,v) = \tfrac{1}{2}\cdot 2 + \tfrac{1}{2}\left(2 + H(u,v)\right),$$

from which it again follows that $H(u,v) = 4$.

Returning to the general set-up, for each vertex v, define $M[v]$ to be the probability matrix M with its v-th row and column deleted, and define $T[v]$ to be the v-th column of M with its v-th row deleted.

Theorem 3.4. For each pair of vertices $u \neq v$,

$$H(u,v) = \left((I_{p-1} - M[v])^{-2}\, T[v]\right)_u .$$

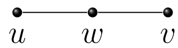
Proof. Let G' be the graph formed from G by removing the vertex v and all of its incident edges. Define the weight of an edge a, b in G' by $\mathrm{wt}(a,b) := M_{ab}$. Then $M[v]$ is a weighted adjacency matrix for G', and $(M[v]^\ell)_{a,b}$ is the sum of the weight of the ℓ-walks starting at a and ending at b. It coincides with the probability of a walk of length ℓ from a to b in G that never passes through v.

It turns out the formula $\sum_{n\geq}(n+1)x^n = 1/(1-x)^2$, which holds for real numbers x with norm less than 1, also holds when x is the matrix $M[v]$. A proof relying on the Perron-Frobenius Theorem is given in our text. The probability of moving from vertex w to vertex v in one step is $\mu(w,v)/d_w$, which is the w-th entry of $T[v]$. Therefore, summing over $w \neq v$,

$$\begin{aligned}
H(u,v) &= \sum_w \sum_{n\geq 0} (n+1)(M[v]^n)_{uw} \frac{\mu(w,v)}{d_w} \\
&= \sum_w \frac{\mu(w,v)}{d_w} \sum_{n\geq 0} (n+1)(M[v]^n)_{uw} \\
&= \sum_w \frac{\mu(w,v)}{d_w} \left(\sum_{n\geq 0} (n+1)(M[v]^n) \right)_{uw} \\
&= \sum_w \frac{\mu(w,v)}{d_w} \left((I_{p-1} - M[v])^{-2}\right)_{uw} \\
&= \left((I_{p-1} - M[v])^{-2} T[v]\right)_u
\end{aligned}$$

□

Example. Consider our earlier example $G = P_3$:



We have

$$M = \begin{array}{c c} & \begin{array}{ccc} u & w & v \end{array} \\ \begin{array}{c} u \\ w \\ v \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix} \end{array}, \quad M[v] = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{pmatrix}, \quad T[v] = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}.$$

Therefore,

$$(I_2 - M[v])^{-2} = \begin{pmatrix} 1 & -1 \\ -\frac{1}{2} & 1 \end{pmatrix}^{-2} = \left(\begin{pmatrix} 1 & -1 \\ -\frac{1}{2} & 1 \end{pmatrix}^{-1} \right)^2 = \left(2 \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & 1 \end{pmatrix} \right)^2 = \begin{pmatrix} 6 & 8 \\ 4 & 6 \end{pmatrix},$$

and

$$(I_2 - M[v])^{-2} T[v] = \begin{pmatrix} 6 & 8 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

It follows that $H(u, v) = 4$ and $H(w, v) = 3$.