Math 372 Homework for Wednesday, Week 14

Problem 1. Let C_n be the cycle graph with vertices $v_1, \dots, v_n$ and edges $\{v_i, v_{i+1}\}$ for all i (with indices modulo n so they remain in the set $[n]$). This problem outlines a proof that $\operatorname{Jac}(C_n) \simeq \mathbb{Z}/n\mathbb{Z}$.

First note that $\{v_i - v_n : i = 1, \dots, n-1\}$ generates

$$\mathbb{Z}V_0 := \left\{ \sum_{i=1}^{n} a_i v_i \in \mathbb{Z}V : \sum_{i=1}^{n} = 0 \right\}.$$

(Equivalently, $(1,0,\dots,0,-1), (0,1,0,\dots,0,-1), \dots, (0,\dots,0,1,-1)$ generate $(\mathbb{Z}^n)_0$, the group of integer vectors whose coordinates sum to 0.) It follows that the equivalence classes of the $v_i - v_n$ generate the $\operatorname{Jac}(G) := \mathbb{Z}V_0/\operatorname{im}(L)$, i.e., every element of $\operatorname{Jac}(G)$ is an integer linear combination of the $v_i - v_n$.

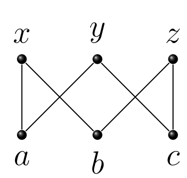
(a) Prove that $k(v_1 - v_n) = v_k - v_n \bmod \operatorname{im}(L)$, i.e., $k(v_1 - v_n) = v_k - v_n$ in $\operatorname{Jac}(G)$, for $k = 1, \dots, n-1$, by finding vectors σ_k such that $k(v_1 - v_n) - L\sigma_k = v_k - v_n$. (Hint: to discover a correct general form for σ_k, you might warm up with the case $n = 6$, a hexagon. Start with $kv_1 - kv_6$ for various k, and perform set-firings to get $v_k - v_6$. Then tally up how many times each vertex was fired to find σ_k.)

(b) After the first part of this problem, we know that $\operatorname{Jac}(G)$ is generated by $v_1 - v_n$. So it is a cyclic group, isomorphic to $\mathbb{Z}/m\mathbb{Z}$ for some m. It just remains to be shown that $m = n$. In this part, we provide a tool for doing so.

Let $\eta := (1, 2, \dots, n) \in \mathbb{Z}^n$, and let $\delta \in (\mathbb{Z}^n)_0$. Show that if $\delta = 0 \bmod \operatorname{im}(L)$ then $\delta \cdot \eta = 0 \bmod n$. (Note: the condition on δ says it is a integer linear combination of the columns of the Laplacian matrix, and the thing you are asked to prove is linear in δ.)

(c) Use the second part of this problem to show that $k(v_1 - v_n)$ are distinct as elements of $\operatorname{Jac}(C_n)$ for $k = 0, 1, \dots, n-1$, and hence that that $\operatorname{Jac}(C_n) \simeq \mathbb{Z}/n\mathbb{Z}$.

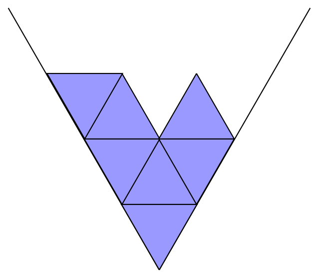
Problem 2. Consider the *crown poset*, $\mathcal{C}_3$:



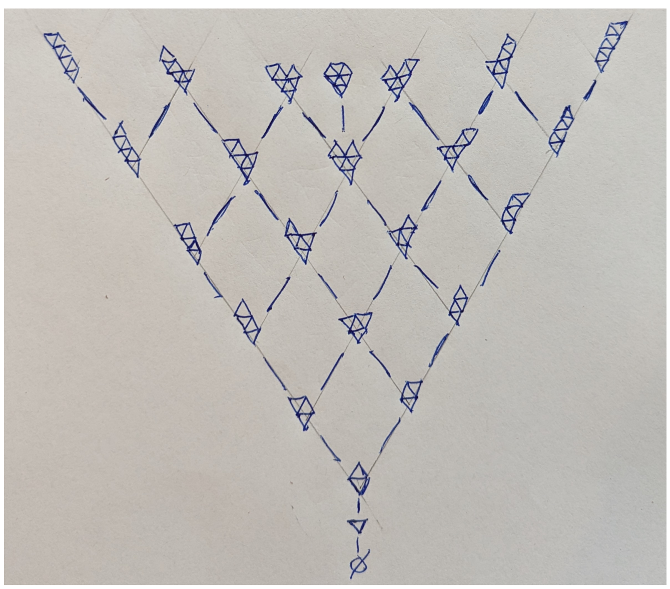
(a) Draw the Hasse diagram for $J(\mathcal{C}_3)$, the distributive lattice of order ideals for $\mathcal{C}_3$, circling the join irreducible elements.

(b) Draw a divisor on a graph whose firing lattice is $J(\mathcal{C}_3)$.

Problem 3. Consider dropping equilateral triangles into a V-shape:



Analogous to Young's lattice, we get a distributive lattice T:



(a) Describe the join irreducibles of T.

(b) Draw the Hasse diagram of the subposet of T consisting of the join irreducibles.

(c) [Challenge] From the Hasse diagram for the join irreducibles of T, we could create a graph/divisor pair whose firing graph is T. Can you find a simpler graph/divisor pair whose firing graph is T? (This would be analogous to the infinite path graph divisor we used for Young's diagram.)