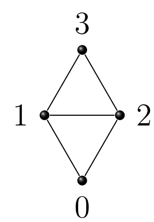
Math 372 Homework for Wednesday, Week 12

Problem 1. Let $n \geq 4$, and let G be the complete graph K_n with one edge removed (so there will be a total of $\binom{n}{2} - 1$ edges). Find the structure of $\mathrm{Jac}(G)$ as a finitely generated abelian group, i.e., find $n_i > 1$ and k such that

$$\mathrm{Jac}(G) \simeq \oplus_{i=1}^{k} \mathbb{Z}/n_i\mathbb{Z},$$

by applying integer row and column operations to the reduced Laplacian of G with respect to any vertex (it won't matter which you choose). For your proof, once you find a general sequence of operations leading to a diagonal matrix, you can opt to explain the sequence by example for the case $n = 7$. However, it should be clear that your sequence of moves would work for any $n \geq 4$. Also, you should state the structure for general n.

Problem 2. Consider the diamond graph G with vertices $\{0, 1, 2, 3\}$ with sink vertex 0 as pictured below:



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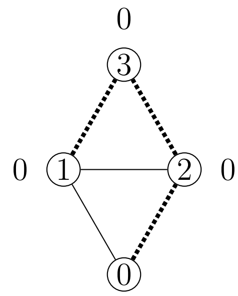
Find the 8 superstable configurations of G (with respect to the sink vertex 0). For each of these superstables c, let $T(c)$ be the corresponding spanning tree produced by the depth-first search burning algorithm. Verify that the κ-inversion number of $T(c)$ is equal to $g - \deg(c)$ where g is the dimension of the cycle space for G. (In this case, you should find that the inversion number for each tree is equal to its κ-inversion number. Thus, there is no difference between inversions and κ-inversions for G. This turns out to be the case for any *threshold graph*, of which G is an example).

Here is an example. The superstable $c = (0, 0, 0, 0)$ is shown below along with its corresponding depth-first search spanning tree, the κ-inversions, and the verification that the number of κ-inversions is $g - \deg(c)$:



κ-inversions (2,1),(3,1)

$2 = 2 - \deg(c) = 2 - 0$