Math 372 Homework for Wednesday, Week 11

Problem 1. Let K_4 be the complete graph on the vertex set $\{1, 2, 3, 4\}$, and let T be the spanning tree with edge set $\left\{\overline{12}, \overline{13}, \overline{14}\right\}$.

(a) Compute the bases for the cycle space $\mathcal{C}$ and the cut space $\mathcal{C}^*$ corresponding to T.

(b) Choose one of your basis elements from $\mathcal{C}$ and one from $\mathcal{C}^*$, and verify that they are orthogonal.

Problem 2. Let $G = (V, E)$ be an undirected, connected multigraph.

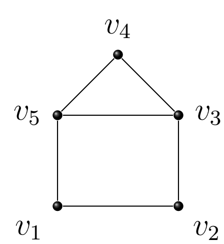
(a) Let $\emptyset \subsetneq U \subsetneq V$. Prove that the cut $c_U^* \in \mathbb{Z}E$ is in $\mathrm{Span}_{\mathbb{Z}} \{c_v^* : v \in V\}$ where c_v^* is the vertex cut corresponding to v.

(b) Let L be the Laplacian of G. Show that

$$\partial(\mathcal{C}^*) = \mathrm{im}(L).$$

(Hint: use part (a).)

Problem 3. Consider the *house graph* H displayed below:



Determine the structure of $\mathrm{Jac}(H)$ by computing the Smith normal form for the reduced Laplacian of H with respect to vertex v_1 by hand. In other words, find $d_i \in \mathbb{Z}$ such that $\mathrm{Jac}(H) = \oplus_{i=1}^{k} \mathbb{Z}/d_i\mathbb{Z}$. (Note: you can omit terms where $d_i = 1$ since $\mathbb{Z}/1\mathbb{Z} = \{0\}$).

Problem 4. Determine the structure of $\mathrm{Jac}(K_n)$ for the complete graph K_n by computing the Smith normal form for the reduced Laplacian of K_n with respect to any vertex.