Math 372 Homework for Wednesday, Week 9

Let P be a locally finite poset. Recall that we defined the incidence algebra $\mathcal{I}(P)$ for P. Its elements are functions from the set of intervals to a fixed field K, and the product is convolution: for $x \leq y$,

$$(fg)(x,y) := \sum_{x \leq z \leq y} f(x,z)g(z,y).$$

The identity for $\mathcal{I}(P)$ is

$$\delta(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

The zeta function for P is $\zeta(x,y) = 1$ for all $x \leq y$. The Möbius function, μ, for P which we defined in classes turns out to be the multiplicative inverse of ζ.

PROBLEM 1. (Interesting observation.) For each interval $[x,y]$ of P, let $\overline{[x,y]} \in \mathcal{I}(P)$ denote the corresponding characteristic function:

$$\begin{aligned} \overline{[x,y]}\colon \mathrm{Int}(P) &\to K \\ [s,t] &\mapsto \begin{cases} 1 & \text{if } s = x \text{ and } t = y \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Then **every element of $\mathcal{I}(P)$ can be thought of as an infinite linear combination of these characteristic functions of intervals**. (Since P is locally finite, its value on any particular interval will only involve summing a finite number of terms.)

Task: Given intervals $[x,y]$ and $[s,t]$, compute the product (convolution) of their corresponding characteristic functions, and express your solution in terms of characteristic functions. (You should find a very nice, intuitive rule.)

One could then define the product for $\mathcal{I}(P)$ by using the simple rule you have just found for characteristic functions, and then extending bilinearly.

PROBLEM 2. Define

$$\eta(x,y) = \begin{cases} 1 & \text{if } y \text{ covers } x \\ 0 & \text{otherwise.} \end{cases}$$

Recall that y covers x if $x < y$ and there is no z such that $x < z < y$.

Also recall that a *chain* in P of length n is a sequence of elements $x_0 < x_2 < \cdots < x_n$. A *maximal chain* (of length n) is a chain $x_0 < x_1 \cdots < x_n$ where x_i covers x_{i-1} for all i (so the chain cannot be made longer). A *multichain* of length n is a sequence of elements $x_0 \leq x_1 \leq \cdots \leq x_n$.

(a) What does $\zeta^2(x, y)$ count? What does $\zeta^k(x, y)$ count?

(b) What does $(\zeta - \delta)(x, y)$ count? What does $(\zeta - \delta)^k(x, y)$ count?

(c) What does $\eta^k(x, y)$ count?

(d) Prove that $\delta - \eta$ has a multiplicative inverse. (One problem you will need to address is why the infinite series that will occur to you is actually summable.)

(e) What does $(\delta - \eta)^{-1}(x, y)$ count?

Problem 3. Use the principle of inclusion-exclusion to find a formula for the number of surjective functions $f\colon [n] \to [n]$. (Hint: for each $i \in [n]$, you may want to consider those functions that do not contain i in their image.)

Problem 4. How many permutations of the 26 letters of the English alphabet contain at least one of the strings *rain*, *storming*, *hot*, or *cold*? (Careful: no letter may be used more than once.) Use the principle of inclusion-exclusion to solve the problem, and then use a computer to find the decimal expansion. (I think there are 24 digits, with largest digit equal to 6.)