Math 372 Homework for Wednesday, Week 8

Derangements. Recall that in class we showed that the number of derangements of n objects is

$$D_n = n! \sum_{i=0}^{n} (-1)^i \frac{1}{i!}.$$

and the exponential generating function for the D_n is

$$D = \sum_{n=0}^{\infty} D_n \frac{x^n}{n!} = \frac{e^{-x}}{1-x}.$$

Problem 1. Ten people write their names on pieces of paper, mix them up in a bag, and then draw them out again at random. What is the probability that exactly six people get their own names back again? (Assume each person has a distinct name.)

Problem 2. Show that for $n > 0$,

$$D_n = \left\lceil \frac{n!}{e} \right\rfloor,$$

the closest integer to $n!/e$.

Problem 3. Using the closed form for D, show the following identity for the exponential generating function for derangements:

$$(1-x)\frac{dD}{dx} = D - e^{-x}.$$

Problem 4. Differentiating the power series expression for D, use the above to show that

$$D_{n+1} = (n+1)D_n + (-1)^{n+1}.$$

Problem 5. Use the above formula to calculate D_n and calculate decimal approximations for $n!/e$ for $n = 1, 2, 3, 4, 5, 6$. (Recall that $D_0 = 1$.)

Dirichlet series.

PROBLEM 6. Find a closed form for the Dirichlet series for $\{n^2\}$ in terms of the Riemann ζ-function.

PROBLEM 7. Find a closed form for the Dirichlet series for $\{\ln(n)\}$ in terms of the Riemann ζ-function.

PROBLEM 8. **Euler ϕ-function**. For $n \in \mathbb{N}$, let

$$\phi(n) = \#\{k \in \mathbb{N} : 1 \leq k \leq n \text{ and} \gcd(k, n) = 1\}.$$

(a) By considering the reduced forms of the fractions $\{1/n, 2/n, \ldots, n/n\}$, show that

$$n = \sum_{d|n} \phi(d).$$

(b) Let f be a multiplicative function and define $g(n) = \sum_{d|n} f(d)$. Show that g is multiplicative.

(c) Show that ϕ is multiplicative.

(d) If p is prime and $k > 0$, show that $\phi(p^k) = p^{k-1}(p - 1)$.

(e) Let Φ be the Dirichlet series for $\{\phi(n)\}$. Since ϕ is multiplicative, we know from class that Φ can be factored as a product of series, one for each prime. Use this factorization to show that

$$\Phi = \frac{\zeta(s-1)}{\zeta(s)}.$$