Math 372 Homework for Wednesday, Week 7

PROBLEM 1. Let $G = (V, E)$ be an undirected graph with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set E. (For simplicity, we will assume G has no multiple edges.) A *divisor* on G is a vector $D \in \mathbb{Z}^n$. Think of the integer $D(i)$ as a number of dollars assigned to vertex v_i. If $D(i) < 0$, then v_i is in debt. A *lending move* by v_i consists of v_i giving a dollar to each of its neighbors and, consequently, losing $\deg(v_i)$ dollars, itself. (By definition v_j is a *neighbor* of v_i if v_iv_j is an edge of G.) A *borrowing move* by vertex v_i is the opposite: v_i takes a dollar from each of its neighbors. Say divisors D and D' are *linearly equivalent*, denoted $D \sim D'$, if D' can be derived from D through a series of lending and borrowing moves. The *dollar game* on G starts with a divisor D, and the goal is to make lending and borrowing moves to arrive at a linearly equivalent divisor E such that $E \geq 0$, i.e., such that $E(i) \geq 0$ for $i = 1, \ldots, n$.

(a) Use the Polya counting technique to compute the number of necklaces with 4 blue beads and 3 red beads.

(b) Let $G = C_3$, the cycle graph on 3 vertices v_1, v_2, v_3 (a triangle), and consider the divisor $D = (4, 0, 0)$. By performing lending and borrowing moves, determine all divisors $E \geq 0$ such that $D \sim E$. In other words, find all possible winning end conditions for the dollar game on G starting at D.

(c) Give a combinatorial bijection between the necklaces you found in (a) and the divisors E you found in (b). To give a "combinatorial bijection", you should find a natural and meaningful story that says which necklace should be associated with which winning divisor E. (If this is done correctly, it will generalize to necklaces with k blue beads and ℓ red beads whenever $\gcd(k, \ell) = 1$.)

PROBLEM 2.

(a) Consider a power series of the form $f = 1 + 2x + 5x^2 + 3x^3 + \cdots$ Since you do not know all of the coefficients of f, you cannot compute all the coefficients of $1/f$, however determine as many as possible.

(b) Let $f = x + x^2$ and $f^{-1} = \sum_{n \geq 1} a_n x^n$, so that we have

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x.$$

Find a_n for $n = 1, 2, 3, 4$.

PROBLEM 3. Find closed forms for the ordinary power series for the following sequences:

(a) $\{\sum_{k=0}^{n}(k+2k^3)\}_n = \{0, 3, 21, 78, 210, \dots\}$.

(b) The odd integers, $\{1, 3, 5, \dots\}$.

(c) $\{1, 0, 1, 0, 1, 0, \dots\}$.

(d) $\{1, 0, 2, 0, 3, 0, 4, 0, \}$.

PROBLEM 4. Fix $n \geq 0$. Find a closed form for the ordinary power series for the sequence $\{\sum_{i=0}^{k}(-1)^i\binom{n}{i}i\}_k$. Compare coefficients to derive an identity for binomial coefficients.

PROBLEM 5. Consider the Fibonacci numbers, $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1}+F_{n-2}$ for $n \geq 2$. Let F be the ordinary generating function for the Fibonacci numbers.

(a) Use the fact that $F = x/(1-x-x^2)$ to write the multiplicative inverse of $1+F$ as a power series G. Determine all of the coefficients of G explicitly.

(b) Equating the coefficient of x^n on both sides of $(1+F)G = 1$ yields which identities among Fibonacci numbers?