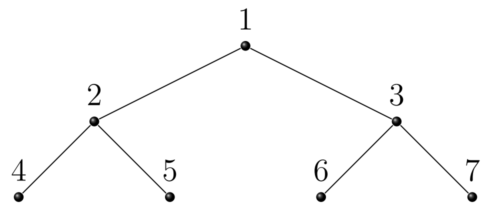
Math 372 Homework for Wednesday, Week 6

Problem 1. For a simple graph Γ with vertex set V, we can define an automorphism of Γ to be a bijection $\phi\colon V \to V$ such that u and v are adjacent if and only if $\phi(u)$ and $\phi(v)$ are adjacent. The automorphisms form a group under composition, called the automorphism group $\mathrm{Aut}(\Gamma)$ of Γ. Let Γ be the graph shown below.



Let G be the automorphism group of Γ, so G has order eight.

(a) What is the cycle index polynomial of G, acting on the vertices of Γ?

(b) In how many ways can one color the vertices of Γ in n colors, up to the symmetry of Γ?

Problem 2. A regular tetrahedron T has four vertices, six edges, and four triangles. The rotational symmetries of T (no reflections allowed) form a group G of order 12.

(a) What is the cycle index polynomial of G acting on the vertices of T?

(b) In how many ways can the vertices of T be colored in n colors, up to rotational symmetry?

Problem 3. Ten balls are stacked in a triangular array with ball 1 at the top, balls 2 and 3 in the next row, etc. (Think of billiards.) The triangular array is free to rotate in two dimensions.

(a) Find the generating function for the number of inequivalent colorings using the ten colors $r_1, r_2, \ldots, r_{10}$. (You don't need to simplify your answer.)

(b) How many inequivalent colorings have four red balls, three green balls, and three chartreuse balls? How many have four red balls, four turquoise balls, and two aquamarine balls?

Problem 4. The dihedral group D_4 of order 8 (generated by a flip along the diagonal and a rotation) acts on the set X of 64 squares of an 8×8 chessboard B. Find the number of ways to choose two subsets $S \subseteq T$ of X, up to the action of D_4 . For instance, all eight ways to choose S to be a single corner square s and T to be $\{s, t\}$, where t is adjacent to s (i.e., has an edge in common with s), belong to the same orbit of D_4. Write your answer as a (short) finite sum.