# Math 372

November 30, 2022

# Today

Goal: Distributive lattices.

# Review

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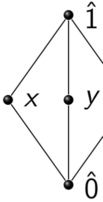
A *lattice* is a poset $L$ in which every pair of elements $x, y \in L$ as a meet and a join.

A lattice $L$ is *distributive* if for all $x, y, z \in L$,

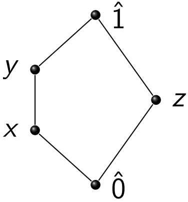
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

# Review

Obstructions to being distributive:




Diamond lattice, $M_3$




Pentagon lattice, $N_5$

**Proposition.** A lattice is distributive if and only if none of its sublattices is isomorphic to $M_3$ or $N_5$.

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The set $J(P)$ of order ideals ordered by set-inclusion is a distributive lattice.

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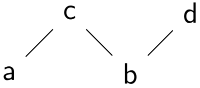
**Example.**




$P$

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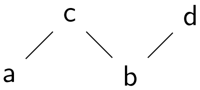


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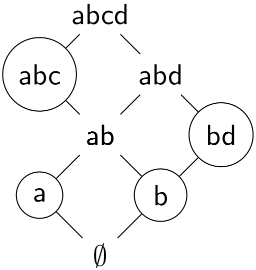


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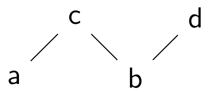


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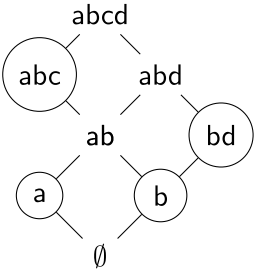


$J(P)$

$x$ in a lattice is *join irreducible* if $x \neq \hat{0}$ and if it is not possible to write $x = y \vee z$ with $y < x$ and $z < x$ ($x$ covers exactly one element).

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**Theorem.** Let $P$ be a poset in which every principal order ideal is finite. Then the poset $J_f(P)$ of *finite* order ideals of $P$, ordered by inclusion, is a finitary distributive lattice. Conversely, if $L$ is a finitary distributive lattice and $P$ is its subposet of join-irreducibles, then every principal order ideal of $P$ is finite, and $L \simeq J_f(P)$.

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Define $D \in \mathrm{Div}(G)$ by

$$D(x) = \begin{cases} 0 & \text{if } x \in \{\alpha, \beta\}, \\ \deg_G(x) & \text{if } x \text{ is a minimal element of } P, \\ n(x) & \text{otherwise.} \end{cases}$$

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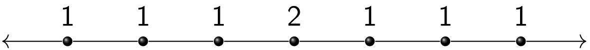
Then the firing graph $\mathcal{F}(D)$ has no cycles, and when considered as a poset, $\mathcal{F}(D) \simeq J_f(P)$.

# Young's lattice

We have seen that Young's lattics is given by the firing graph of the divisor



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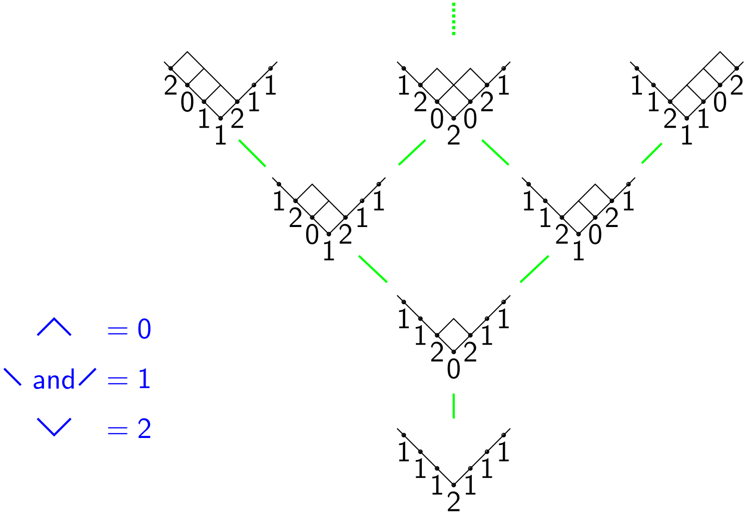

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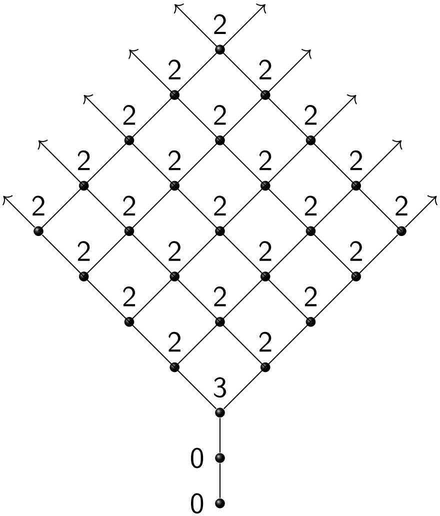
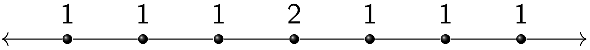
Theorem 1 gives Young's diagram as the firing graph for the divisor pictured on the next slide.

# Young's lattice



# Young's lattice

The divisor just pictured is much more complicated than




Question: Is there a principled way in which we could have found this simpler divisor?

# Generalization of Young's lattice

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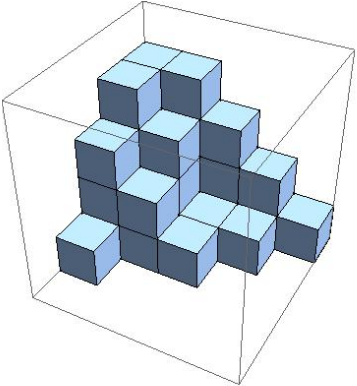
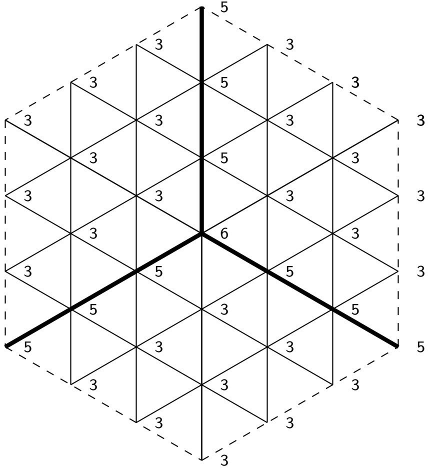
**Example:** $k = 3$ gives plane partitions:



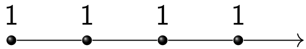
# Plane partition lattice from a divisor




(Again, this does not come from Theorem 1.)

# Lattice of shifted shapes

The firing graph for the divisor

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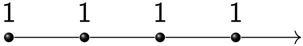


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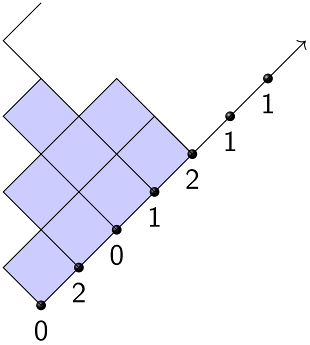
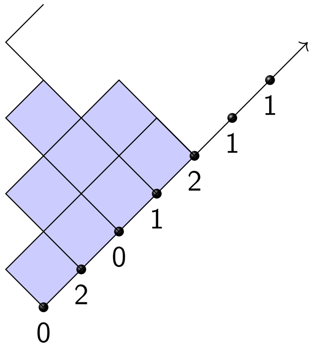


# Lattice of shifted shapes

Question: What is the poset of join irreducibles of the lattice of shifted shapes?

$(4, 3, 1)$



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**Problem 2.** Show that when $\mathcal{F}(D)$ is acyclic it is the Hasse diagram for a lattice.

**Problem 3.** Let $\mathcal{FL}$ denote the collection of all lattices of the form $\mathcal{F}(D)$. We have seen that $\mathcal{FL}$ contains all finitary distributive lattices.

# Locally free lattices

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A *firing script* is a formal sum of vertices: $\sigma = \sum_{i=1}^{n} a_i v_i$. It is *legal* for a divisor $D$ if there is a legal sequence of vertex firings $w_1, \ldots, w_k$ where $k = \sum_{i=1}^{n} a_i$ such that $\sigma = \sum_{i=1}^{k} w_i$.

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**Theorem.** (Björner, Lovàsz, Shor, 1991). Let $G$ be a finite graph, and let $D \in \mathrm{Div}(G)$ be stabilizable. Then the firing graph $\mathcal{F}(D)$ is a locally free lattice. Given $D', D'' \in \mathcal{F}(D)$, let $\sigma'$ and $\sigma''$ be legal firing scripts such that $D \xrightarrow{\sigma'} D'$ and $D \xrightarrow{\sigma''} D''$.
Then $D \xrightarrow{\sigma' \vee \sigma''} D''$, where $(\sigma' \vee \sigma'')(v) := \max\{\sigma'(v), \sigma''(v)\}$ for all vertices $v$ of $G$.

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**Proved earlier:** for each $D' \in \mathcal{F}(D)$, there exist a unique legal firing script $\sigma$ such that $D \xrightarrow{\sigma} D'$ and the support of $\sigma$ does not contain all of the vertices.

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It is *not* generally true that the meet is $D[\sigma \wedge \tau]$.